MATH 2020B

Tutorial 9

Green's Theorem

Let $\Omega \subseteq \mathbb{R}^2$ be open, $\vec{F} = M\vec{i} + N\vec{j}$ be C^1 vector field on Ω, C be a piecewise smooth simple closed anti-clockwise oriented curve enclosing a region R which lies entirely in Ω.

Then,

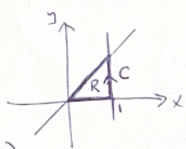
$$\oint_C \vec{F}\cdot\hat{n}\,ds = \oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy$$

$$\oint_C \vec{F}\cdot\hat{T}\,ds = \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

1. Find the counterclockwise circulation and outward flux for the field $\vec{F}$ and curve C.

$\vec{F} = \underbrace{(x+y)}_{M}\vec{i} - \underbrace{(x^2+y^2)}_{N}\vec{j}$

C: The triangle bounded by $y=0,\ x=1,\ y=x$



- Counterclockwise circulation

$$= \oint_C M\,dx + N\,dy$$

$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy \quad \text{(by Green's Theorem)}$$

$$= \int_0^1 \int_0^x (-2x-1)\,dy\,dx = -\frac{7}{6}$$

- Outward flux

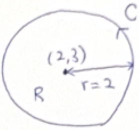
$$= \oint_C M\,dy - N\,dx$$

$$= \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy \quad \text{(by Green's Theorem)}$$

$$= \int_0^1 \int_0^x (1-2y)\,dy\,dx = \frac{1}{6}$$

2. Evaluate the integral $\oint_C (6y+x)\,dx + (y+2x)\,dy$

C : The circle $(x-2)^2 + (y-3)^2 = 4$.



$$\oint_C \underbrace{(6y+x)}_{M}\,dx + \underbrace{(y+2x)}_{N}\,dy$$

$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy \quad \text{(by Green's Theorem)}$$

$$= \iint_R (2-6)\,dx\,dy$$

$$= \iint_R (-4)\,dx\,dy = (-4)\cdot \text{Area of } R$$

$$= (-4)\cdot \pi(2)^2 = -16\pi$$

3. Evaluate the integral $\oint_C 4x^3y\,dx + x^4\,dy$

for any closed path C.

$\vec{F} = 4x^3y\,\vec{i} + x^4\,\vec{j}$ Domain $\Omega = \mathbb{R}^2$

We can apply Green's Theorem for any closed path C.

$$\oint_C 4x^3y\,dx + x^4\,dy$$

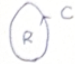
$$= \iint_R \left(\frac{\partial}{\partial x}(x^4) - \frac{\partial}{\partial y}(4x^3y)\right) dx\,dy \quad \text{(by Green's Theorem)}$$

$$= \iint_R (4x^3 - 4x^3)\,dx\,dy$$

$$= \iint_R 0\,dx\,dy = 0$$

4. Find the area of the region enclosed by the curve :

The curve $\vec{r}(t) = (2\cos t)\vec{i} + (3\sin t)\vec{j}$, $0 \le t \le 2\pi$.



$$\boxed{\text{Area of } R = \frac{1}{2}\oint_C x\,dy - y\,dx}$$

$$\| $$

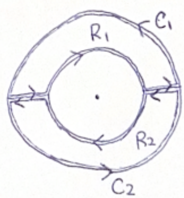
$$\iint_R 1\,dydx$$

$$\|$$

$$\iint_R \left(\tfrac{1}{2}+\tfrac{1}{2}\right) dydx$$

$$\begin{aligned}
\text{Area} &= \frac{1}{2}\oint_C x\,dy - y\,dx \\
&= \frac{1}{2}\int_0^{2\pi} (2\cos t)(3\cos t) - (3\sin t)(-2\sin t)\,dt \\
&= \frac{1}{2}\int_0^{2\pi} 6\cos^2 t + 6\sin^2 t\,dt \\
&= \frac{1}{2}\int_0^{2\pi} 6\,dt = \frac{1}{2}(2\pi)(6) = 6\pi
\end{aligned}$$

5. Evaluate $\oint_C y^3 dx - x^3 dy$. C: two circles of radius 2 and radius 1 centered at the origin with positive orientation.



$$\begin{aligned}
&\oint_C y^3\,dx - x^3 dy \\
&= \oint_{C_1} \cdots\cdots + \oint_{C_2} \cdots\cdots \\
&= \iint_{R_1} -3x^2 - 3y^2\,dydx + \iint_{R_2} -3x^2 - 3y^2\,dydx \\
&= -3\iint_R (x^2+y^2)\,dydx \\
&= -3\int_0^{2\pi}\int_1^2 r^2\cdot r\,dr\,d\theta = -\frac{45\pi}{2}
\end{aligned}$$