MATH 2020B

Tutorial 8

$\vec{F}$ conservative $\quad (\vec{F}$ on $\Omega)$

- $\Rightarrow \frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i} \quad \forall i, j$

$\uparrow$

$\Leftarrow$ (If $\vec{F}$ on a simply connected open domain of $\mathbb{R}^n$)

- $\Leftrightarrow$ admits a differentiable potential function $f: \Omega \to \mathbb{R}$

$(\nabla f = \vec{F})$

$\int_C \vec{F} \cdot \hat{T} \, ds = f(\vec{r}(b)) - f(\vec{r}(a))$ path independent

$= 0$ (if closed curve)

1. $\vec{F} = (e^x \cos y)\vec{i} - (e^x \sin y)\vec{j} + z\vec{k}$

conservative ?

$\vec{F}$ defined at any $(x, y, z) \in \mathbb{R}^3$

domain $= \mathbb{R}^3$ which is simply connected, open.

Let $M = e^x \cos y$, $N = -e^x \sin y$, $L = z$.

$$\frac{\partial M}{\partial y} = -e^x \sin y = \frac{\partial N}{\partial x}$$

$$\frac{\partial M}{\partial z} = 0 = \frac{\partial L}{\partial x}$$

$$\frac{\partial N}{\partial z} = 0 = \frac{\partial L}{\partial y}$$

Therefore, $\vec{F}$ is conservative.

2. Find a potential function f for the field $\vec{F}$.

$$\vec{F} = 2x\,\vec{i} + 3y\,\vec{j} + 4z\,\vec{k}$$

Suppose $\vec{F} = \nabla f$

$$\frac{\partial f}{\partial x} = 2x \Rightarrow f = x^2 + C_1(y,z)$$

$$\frac{\partial f}{\partial y} = 3y \Rightarrow f = \frac{3}{2}y^2 + C_2(x,z)$$

$$\frac{\partial f}{\partial z} = 4z \Rightarrow f = 2z^2 + C_3(x,y)$$

$$f = x^2 + \frac{3}{2}y^2 + 2z^2 + C$$

($\frac{\partial f}{\partial x} = 2x \Rightarrow f = x^2 + g(y,z) + C_1$

$\Rightarrow \quad 3y = \frac{\partial f}{\partial y} = \frac{\partial g}{\partial y} \Rightarrow g(y,z) = \frac{3}{2}y^2 + h(z) + C_2$

$\Rightarrow \quad 4z = \frac{\partial f}{\partial z} = \frac{\partial h}{\partial z} \Rightarrow h(z) = 2z^2 + C_3$

$f = x^2 + \frac{3}{2}y^2 + 2z^2 + C$.)

3. Find a potential function for $\vec{F}$.

$$\vec{F} = (e^x \ln y)\,\vec{i} + \left(\frac{e^x}{y} + \sin z\right)\vec{j} + (y\cos z)\,\vec{k}$$

Suppose $\vec{F} = \nabla f$

$$\frac{\partial f}{\partial x} = e^x \ln y \quad \Rightarrow \quad f = e^x \ln y + C_1(y,z)$$

$$\frac{\partial f}{\partial y} = \frac{e^x}{y} + \sin z \quad \Rightarrow \quad f = e^x \ln y + y\sin z + C_2(x,z)$$

$$\frac{\partial f}{\partial z} = y\cos z \quad \Rightarrow \quad f = y\sin z + C_3(x,y)$$

$$f = e^x \ln y + y\sin z + C$$

4. Show that this integral does not depend on the path taken from A to B.

$$\int_A^B \frac{x\,dx + y\,dy + z\,dz}{\sqrt{x^2+y^2+z^2}}$$

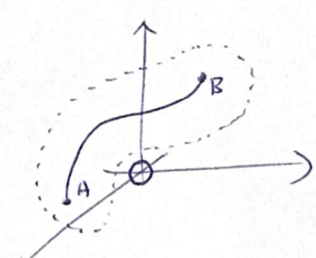
Let $M = \frac{x}{\sqrt{x^2+y^2+z^2}}$, $N = \frac{y}{\sqrt{x^2+y^2+z^2}}$, $L = \frac{z}{\sqrt{x^2+y^2+z^2}}$

$$\frac{\partial M}{\partial y} = \frac{-xy}{(x^2+y^2+z^2)^{\frac{3}{2}}} = \frac{\partial N}{\partial x}$$

$$\frac{\partial M}{\partial z} = \frac{-xz}{(x^2+y^2+z^2)^{\frac{3}{2}}} = \frac{\partial L}{\partial x}$$

$$\frac{\partial N}{\partial z} = \frac{-yz}{(x^2+y^2+z^2)^{\frac{3}{2}}} = \frac{\partial L}{\partial y}$$

$\therefore$ path independent



$\mathbb{R}^3 \setminus \{(0,0,0)\}$ simply connected, open.

$$\frac{x\,dx + y\,dy + z\,dz}{\sqrt{x^2+y^2+z^2}}$$ exact differential form

5. Evaluate the integral

$$\int_{(1,1,2)}^{(3,5,0)} yz\,dx + xz\,dy + xy\,dz$$

(let $M = yz$, $N = xz$, $L = xy$.

$$\frac{\partial M}{\partial y} = z = \frac{\partial N}{\partial x}$$

domain $\mathbb{R}^3$

$$\frac{\partial M}{\partial z} = y = \frac{\partial L}{\partial x}$$

simply connected, open

$$\frac{\partial N}{\partial z} = x = \frac{\partial L}{\partial y}$$

$\therefore \vec{F} = yz\,\vec{i} + xz\,\vec{j} + xy\,\vec{k}$ is conservative.)

Suppose $\vec{F} = \nabla f$

We have $f(x,y,z) = xyz + C$. domain $\mathbb{R}^3$.

Then, $\int_{(1,1,2)}^{(3,5,0)} yz\,dx + xz\,dy + xy\,dz$

$= f(3,5,0) - f(1,1,2)$

$= 0 - 2$

$= -2$