# LECTURE 22

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The exterior derivative $d$ is a fundamental operator in differential geometry that generalizes the concept of differentiation to differential forms. For a $k$-form $\omega$, the exterior derivative $d\omega$ yields a $(k+1)$-form, encoding infinitesimal variations of $\omega$ in all directions. It satisfies the key properties of linearity, the graded Leibniz rule $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{\deg \omega} \omega \wedge d\eta$. Moreover, we set $d(dx_i) = d^2 x_i = 0$ for each $i$.

**Proposition 1.** *If $\omega$ is a differential $k$-form whose components have continously differentiable derivatives, then $d^2\omega = 0$.*

*Proof.* For a function $f : \Omega \to \mathbb{R}$ which is smooth or at least has continuous second derivative, the exterior derivative $df$ is given by:

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i.$$

Applying $d$ again yields:

$$d(df) = d\left(\sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i\right) = \sum_{i=1}^{n} d\left(\frac{\partial f}{\partial x^i}\right) \wedge dx^i = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x^j \partial x^i} dx^j \wedge dx^i.$$

Since mixed partial derivatives commute

$$\frac{\partial^2 f}{\partial x^j \partial x^i} = \frac{\partial^2 f}{\partial x^i \partial x^j}$$

and $dx^j \wedge dx^i = -dx^i \wedge dx^j$, the terms pair up as:

$$\frac{\partial^2 f}{\partial x^j \partial x^i} dx^j \wedge dx^i + \frac{\partial^2 f}{\partial x^i \partial x^j} dx^i \wedge dx^j = \left(\frac{\partial^2 f}{\partial x^j \partial x^i} - \frac{\partial^2 f}{\partial x^i \partial x^j}\right) dx^j \wedge dx^i = 0.$$

Thus, $d^2 f = 0$. For higher-degree forms, the result follows from the graded Leibniz rule and the fact that $d^2$ vanishes on functions. □

Next, we shall explain that Green's theorem, Stokes' theorem (in terms of curl), and divergence theorem all follow from the generalized Stokes' theorem.

**Theorem 2** (Generalized Stokes' Theorem)**.** *Let $M$ be a smooth, oriented $n$-dimensional manifold with boundary in $\mathbb{R}^n$, and let $\partial M$ denote its boundary with the induced orientation. If $\omega$ is a continuously differentiable $(n-1)$-form with compact support on $M$, then*

$$\int_M d\omega = \int_{\partial M} \omega,$$

*where $d\omega$ is the exterior derivative of $\omega$.*

Let me not explain what the general notion of a submanifold is. You only need to know that the curves are 1-dimensional submanifolds, surfaces are 2-dimensional submanifolds, and so on so forth.

To begin, let us look at Green's theorem.

**Example 3.** Let $\omega$ be a 1-form on a region $R \subset \mathbb{R}^2$ given by:

$$\omega = M\,dx + N\,dy,$$

where $M, N$ are smooth functions on $R$. The exterior derivative $d\omega$ is computed as:

$$d\omega = dM \wedge dx + dN \wedge dy = \left(\frac{\partial M}{\partial x} dx + \frac{\partial M}{\partial y} dy\right) \wedge dx + \left(\frac{\partial N}{\partial x} dx + \frac{\partial N}{\partial y} dy\right) \wedge dy.$$

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Using the antisymmetry $dx \wedge dx = dy \wedge dy = 0$ and $dy \wedge dx = -dx \wedge dy$, this simplifies to:

$$d\omega = \frac{\partial M}{\partial y} dy \wedge dx + \frac{\partial N}{\partial x} dx \wedge dy = \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx \wedge dy.$$

Therefore, Green's theorem can be understood as the equation

$$\int_{\partial R} \omega = \oint_{\partial R} (M\,dx + N\,dy) = \iint_R \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = \int_R d\omega.$$

Before moving on the Stokes and divergence theorems, we need a proposition which explains the flux of a vector field through a surface in terms of integration of a certain 2-form.

**Proposition 4.** *Let $\mathbf{r} : R \to \mathbb{R}^3$ be a parametrized surface $S$, and let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ be a vector field defined on an open region containing $S$. Suppose that $S$ is oriented by the unit vector field*

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|}.$$

*Set $\omega$ to be the 2-form*

$$\omega = M dy \wedge dz - N dx \wedge dz + P dx \wedge dy. \tag{1}$$

*Then*

$$\int_S \mathbf{F} \cdot \mathbf{n} d\sigma = \int_S \omega.$$

*Proof.* Recall that by definition

$$\int_S \omega = \int_R \mathbf{r}^*(\omega).$$

We compute the pullback $\mathbf{r}^*(\omega)$ using the parametrization $\mathbf{r}(u,v) = (x(u,v), y(u,v), z(u,v))$. The differentials become:

$$\begin{aligned} dx &= x_u du + x_v dv \\ dy &= y_u du + y_v dv \\ dz &= z_u du + z_v dv \end{aligned}$$

Substituting these into $\omega$ and expanding, we get:

$$\begin{aligned} \mathbf{r}^*(\omega) &= M(y_u du + y_v dv) \wedge (z_u du + z_v dv) - N(x_u du + x_v dv) \wedge (z_u du + z_v dv) + P(x_u du + x_v dv) \wedge (y_u du + y_v dv) \\ &= M(y_u z_v - y_v z_u) du \wedge dv - N(x_u z_v - x_v z_u) du \wedge dv + P(x_u y_v - x_v y_u) du \wedge dv \\ &= \left[ M \frac{\partial(y,z)}{\partial(u,v)} - N \frac{\partial(x,z)}{\partial(u,v)} + P \frac{\partial(x,y)}{\partial(u,v)} \right] du \wedge dv \end{aligned}$$

where the Jacobian determinants appear as coefficients.

Notice that

$$\mathbf{r}_u \times \mathbf{r}_v = \left( \frac{\partial(y,z)}{\partial(u,v)}, -\frac{\partial(x,z)}{\partial(u,v)}, \frac{\partial(x,y)}{\partial(u,v)} \right).$$

Therefore,

$$\mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) = M \frac{\partial(y,z)}{\partial(u,v)} - N \frac{\partial(x,z)}{\partial(u,v)} + P \frac{\partial(x,y)}{\partial(u,v)}$$

and thus

$$\begin{aligned} \int_S \omega &= \int_R \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) du dv \\ &= \int_R \mathbf{F} \cdot \mathbf{n} \|\mathbf{r}_u \times \mathbf{r}_v\| du dv \\ &= \int_S \mathbf{F} \cdot \mathbf{n} d\sigma \end{aligned}$$

which completes the proof. □

**Example 5.** Let $\omega$ be a 2-form on a region $R \subset \mathbb{R}^3$ with coordinates $(x,y,z)$, given by:

$$\omega = M\,dy \wedge dz + N\,dz \wedge dx + P\,dx \wedge dy,$$

where $M,N,P$ are smooth functions on $R$. The exterior derivative $d\omega$ is computed as:

$$d\omega = dM \wedge dy \wedge dz + dN \wedge dz \wedge dx + dP \wedge dx \wedge dy.$$

Expanding each term using

$$dM = \frac{\partial M}{\partial x}dx + \frac{\partial M}{\partial y}dy + \frac{\partial M}{\partial z}dz$$

(and similarly for $dN, dP$) and simplifying via the wedge product's antisymmetry (e.g., $dx \wedge dx = 0$, $dy \wedge dz \wedge dy = 0$, etc.), we obtain:

$$d\omega = \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}\right) dx \wedge dy \wedge dz.$$

By Stokes' Theorem (general form), we have:

$$\int_{\partial R} \omega = \int_R d\omega = \int_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}\right) dx \wedge dy \wedge dz.$$

Rewriting in classical notation (where $dx \wedge dy \wedge dz$ corresponds to the volume element $dV$ and $\omega$ corresponds to the flux form $\mathbf{F} \cdot \mathbf{n} d\sigma$ for $\mathbf{F} = (M,N,P)$), we recover the Divergence Theorem:

$$\iint_{\partial R} \mathbf{F} \cdot \mathbf{n} d\sigma = \iiint_R (\nabla \cdot \mathbf{F})\, dV.$$

Thus, the Divergence Theorem is the special case of Stokes' Theorem applied to a 2-form in $\mathbb{R}^3$.

**Example 6.** Finally, let us recover the classical Stokes' theorem in terms of curl. Let $S \subseteq \mathbb{R}^3$ be a smooth surface with smooth closed boundary curve $\partial S$. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ be a continuously differentiable vector field defined on an open region containing $S$. This time, we attach to $\mathbf{F}$ the 1-form

$$\omega = Mdx + Ndy + Pdz.$$

At this point, you should have no difficulty computing

$$d\omega = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right) dy \wedge dz - \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right) dx \wedge dz + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx \wedge dy.$$

Applying the generalized Stokes' theorem to $\omega$ gives:

$$\int_S d\omega = \int_{\partial S} \omega,$$

which by Proposition 4 translates to

$$\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{r},$$

where $\mathbf{n}$ is the unit normal to $S$ and $d\mathbf{r}$ is the line element along $\partial S$, as desired.