LECTURE 21

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Example 1. Given the vector field

$$\mathbf{F} = \frac{1}{\rho^3}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \text{ where } \rho = \sqrt{x^2+y^2+z^2},$$

we compute the net outward flux across the sphere $\rho = a$.

The outward normal vector field is given by

$$\mathbf{n} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\rho}.$$

Therefore,

$$\mathbf{F} \cdot \mathbf{n} = \frac{x^2+y^2+z^2}{\rho^4} = \frac{\rho^2}{\rho^4} = \frac{1}{\rho^2} = \frac{1}{a^2}$$

$$\text{Flux} = \iint_{\rho=a} \mathbf{F} \cdot \mathbf{n}\, d\sigma = \frac{1}{a^2} \cdot 4\pi R^2 = 4\pi.$$

Note that the answer is independent of a. From this you quickly infer that the outward flux through the sphere $\rho = b$ for any $b > a$ is also 4π, and the outward flux through the region Ω given by $a \leq \rho \leq b$ is $4\pi - 4\pi = 0$. Note that when the sphere $\rho = a$ is viewed as the inner boundary of Ω, its normal vector field should be pointing inward, not outward as before, because now you are observing from inside of Ω, hence the minus sign as in $4\pi - 4\pi$.

On the other hand, we can also understand this in terms of the divergence theorem. A quick calculation shows that the divergence $\nabla \cdot \mathbf{F}$ is 0 on $\mathbb{R}^3 \smallsetminus \{(0,0,0)\}$, which in particular contains Ω. Therefore, by the divergence theorem we also get that the outward flux of $\mathbf{F}$ is 0. Note however that $\mathbf{F}$ is not defined at the origin, so that you cannot apply divergence theorem to conclude that the outward flux through the sphere $\rho = a$ (which indeed contains the origin) is also 0.

Example 2. The electric flux through any closed surface $\partial\Omega$ is proportional to the total charge Q_{enc} enclosed within the surface:

$$\iint_{\partial\Omega} \mathbf{E} \cdot dA = \frac{Q_{\text{enc}}}{\varepsilon_0} \tag{1}$$

where:

- $\mathbf{E}$ is the electric field vector
- dA is the differential area element vector (pointing outward normal to the surface)
- Q_{enc} is the total charge enclosed within the volume Ω
- ε_0 is the vacuum permittivity

Let us consider the case when we have point charges. The Coulomb's law tells us that

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\mathbf{r} \tag{2}$$

where $\mathbf{r}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

Suppose that we have an arbitrary closed surface enclosing the point charge. Then we can always draw a small ball B around the charge which is enclosed by the surface. Then it is not hard to check (1) for B. Now we notice that $\nabla\mathbf{F} = 0$ away from the charge. In particular, the divergence is zero for the region outside of B but inside of Ω. Therefore, the surface integral does not change when we replace B by Ω.

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The content below will not appear on your final, but I think it is something very helpful to know before you leave the course.

Let $x_1,\cdots,x_n$ be a coordinate system on $\mathbb{R}^n$ and let $\Omega \subseteq \mathbb{R}^n$ be an open region. A differential 1-form α is a linear combination of the form

$$\sum_{i=1}^n f_i dx_i, \text{ where } f_i : \Omega \to \mathbb{R} \text{ is a function.}$$

We say that α is smooth or continuously differentiable if each component f_i has this property.

Given two 1-forms $\alpha = \sum_{i=1}^n f_i dx_i$ and $\beta = \sum_{i=1}^n g_i dx_i$, we define the *wedge product* by

$$\alpha \wedge \beta = \sum_{i,j=1}^n f_i g_j dx_i \wedge dx_j$$

subject to the rules that $dx_i \wedge dx_j = -dx_j \wedge dx_i$, in particular, $dx_i \wedge dx_i = 0$ for every i.

A 2-form is a linear combination of the wedge products of two 1-forms. More precisely, a 2-form ω looks like

$$\sum_{1\le i<j\le n} f_{i,j} dx_i \wedge dx_j, \text{ where } f_{i,j} : \Omega \to \mathbb{R} \text{ is a function.}$$

Similarly, a 3-form looks like

$$\sum_{1\le i<j<k\le n} f_{i,j,k} dx_i \wedge dx_j \wedge dx_k,$$

and so on so forth. The wedge product of forms is defined as extending the rule $dx_i \wedge dx_j = -dx_j \wedge dx_i$ in the obvious way.

Wedge products of forms are associative, which means that if α, β, γ are forms, then

$$(\alpha\wedge\beta)\wedge\gamma = \alpha\wedge(\beta\wedge\gamma).$$

Now, let us multiply a 1-form with a 2-form

$$\begin{aligned}
&(x\,dx + xy\,dy)\wedge(e^x\,dx\wedge dy + \sin(z)\,dx\wedge dz)\\
&= x\,dx\wedge e^x\,dx\wedge dy + x\,dx\wedge \sin(z)\,dx\wedge dz\\
&\quad + xy\,dy\wedge e^x\,dx\wedge dy + xy\,dy\wedge\sin(z)\,dx\wedge dz\\
&= xe^x\,dx\wedge dx\wedge dy + x\sin(z)\,dx\wedge dx\wedge dz\\
&\quad + xye^x\,dy\wedge dx\wedge dy + xy\sin(z)\,dy\wedge dx\wedge dz\\
&= -xy\sin(z)\,dx\wedge dy\wedge dz
\end{aligned}$$

Whenever you see terms like $dx\wedge dx$, you set it to 0. For example, $dy\wedge dx\wedge dy = -dx\wedge dy\wedge dy = 0$.

A fundamental operation you can do to a differential form is "*pullback*". Instead of giving you the most formal definition, let me illustrate how to compute this in practice. Suppose that you have a function $\mathbf{r} : R \to \Omega$ for some open $R \subseteq \mathbb{R}^m$. Say the coodinates of $\mathbb{R}^m$ are given by $y_1,\cdots,y_m$, and $\mathbf{r}$ is given by

$$(y_1,\cdots,y_m) \mapsto (f_1(y_1,\cdots,y_m),\cdots,f_n(y_1,\cdots,f_m)),$$

then for any function $g : \Omega \to \mathbb{R}$, the pullback $\mathbf{r}^*(g)$ of g is just the composition $g\circ\mathbf{r}$, and the pullback $\mathbf{r}^*(dx_i)$ of dx_i is just

$$df_i = \sum_{j=1}^m \frac{\partial f_i}{\partial y_j} dy_j.$$

Example 3. Let $\mathbf{r} : \mathbb{R}^2 \to \mathbb{R}^3$ be a smooth surface given by:

$$\mathbf{r}(u,v) = (x(u,v), y(u,v), z(u,v)) = (uv, u+v, \sin u).$$

Let ω be a differential 1-form on Ω:

$$\omega = y\,dx + x\,dy + dz.$$

To pullback ω, you first compute the differentials.

$$dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv = v\,du + u\,dv,$$

$$dy = \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv = du + dv,$$

$$dz = \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv = \cos u\, du.$$

Then, substitute into ω.

$$\mathbf{r}^* \omega = (u+v)(v\,du + u\,dv) + uv(du+dv) + \cos u\,du = \mathbf{r}^*\omega = (2uv + v^2 + \cos u)\,du + (u^2 + 2uv)\,dv.$$

Roughly speaking, a 1-form is an object that can be naturally integrated over a curve, a 2-form over a surface, and so on for higher-dimensional forms. The base case is that, if $(x_1, \cdots, x_n)$ is a coordinate system on $\mathbb{R}^n$, then for any region $\Omega \subseteq \mathbb{R}^n$, we shall define

$$\int_\Omega f(x_1, \cdots, x_n) dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n = \int_\Omega f.$$

By the very nature of wedge products and pullback, the integration of differential forms is independent of the choice of coordinates (up to orientation). For example, suppose that we have open regions R and R' with coordinates (u,v) and (x,y) respectively, and there is a 2-form $\omega = f(x,y) dx \wedge dy$. Then for any bijection $g : (u,y) \mapsto (x,y)$ which is continuously differentiable with its inverse, we have

$$\int_R g^*\omega = \pm \int_{R'} \omega.$$

depending on whether g preserves orientation or not. To see why this recovers the change of variable formula for integrals in terms of Jacobians, we compute.

Roughly speaking, differential forms are objects designed for integration over manifolds in a coordinate-independent way. Specifically:

- A 1-**form** can be naturally integrated over a *curve*,
- A 2-**form** over a *surface*,
- And more generally, a k-**form** over a *k-dimensional manifold*.

The Base Case: Integration on $\mathbb{R}^n$. Let $(x_1, \ldots, x_n)$ be a coordinate system on $\mathbb{R}^n$, and let $\Omega \subseteq \mathbb{R}^n$ be a region. The integral of a smooth function f weighted by the canonical volume form is defined as:

$$\int_\Omega f(x_1, \ldots, x_n)\, dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n := \int_\Omega f\, dV,$$

where dV is the standard volume element. This aligns with classical integration when Ω is oriented positively.

Coordinate Independence and Pullbacks. A key property of differential forms is that their integrals are *coordinate-independent* (up to orientation), thanks to the antisymmetry of the wedge product and the behavior of pullbacks.

Example 4. Consider open regions $R \subset \mathbb{R}^2$ with coordinates (u,v) and $R' \subset \mathbb{R}^2$ with coordinates (x,y). Let $\omega = f(x,y)\, dx \wedge dy$ be a 2-form on R', and let $g\colon R \to R'$ be a diffeomorphism (i.e., a bijection that is continuously differentiable with its inverse). The pullback of ω under g satisfies:

$$\int_R g^*\omega = \pm \int_{R'} \omega,$$

where the sign depends on whether g preserves $(+)$ or reverses $(-)$ orientation.

This recovers the classical change of variables formula. Explicitly, if $g(u,v) = (x(u,v), y(u,v))$, then:

$$g^*\omega = f(x(u,v), y(u,v)) \left(\frac{\partial(x,y)}{\partial(u,v)} \right) du \wedge dv,$$

where $\frac{\partial(x,y)}{\partial(u,v)}$ is the Jacobian determinant. Thus:

$$\int_{R'} f(x,y)\, dx \wedge dy = \pm \int_R f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du \wedge dv,$$

matching the standard transformation rule for integrals. The sign is positive if the transformation g sends a counterclockwise circle to a counterclockwise circle, and negative otherwise.