LECTURE 16

ZIQUAN YANG

1. Green's theorem continued

Recall that in a simply connected region Ω, a vector field $\mathbf{F}(x,y) = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$ is conservative if and only if its components satisfy:

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}. \tag{1}$$

Now this is easy to understand using Green's theorem. Indeed, Ω being simply connected implies that for any oriented closed curve C inside Ω, the region enclosed by C inside $\mathbb{R}^2$ completely lies in Ω. The fact roughly says that if a vector field does not have infinitesimal circulation everywhere, then it does not have circulation along any closed curve. Now think about our favorite counterexample when the simply connectedness assumption is dropped:

$$\mathbf{F} = \frac{-y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}. \tag{2}$$

Obviously, you wouldn't think of the vector field as having "no circulation" if you think about what it looks like. However, it does satisfy equation (1). If you draw small circles which do not enclose the origin, then the circulation is zero. However, if you draw any circle centered at the origin, then the circulation you get is a constant, 2π.

Why is the situation for the vector field

$$\mathbf{F} = -y\mathbf{i} + x\mathbf{j} \tag{3}$$

different? This vector field has the same general shape, but the vectors are getting shorter and shorter towards the origin. The smaller the circle you draw, the smaller the circulation you get. This is not the case with (2). No matter how small the circle you draw (around the origin), the circulation is 2π, so for (2) the **circulation density** is infinite at the origin. In general, the circulation density of a vector field is the difference

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}.$$

Let us denote it by $\mathrm{curl}(\mathbf{F})$, because later you see that it is the special case of a more general notion of "curl" of a vector field.

Similarly, the **flux density** is defined to be

$$\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}.$$

It is also called the **divergence**.

Let us remark that it is not the case that you cannot use Green's theorem to analyze the vector field in (2). Let C be any simple closed curve enclosing the origin, say with counterclockwise orientation. Then you can draw a small circle C' around the origin also with counterclockwise orientation, such that C' is enclosed by C. Then Green's theorem implies that for (2)

$$\oint_C \mathbf{F} d\mathbf{r} = \oint_{C'} \mathbf{F} d\mathbf{r}, \tag{4}$$

because $\mathbf{F}$ is defined on region D between C and C'. You can divide D into simply connected regions, and the integrals of $\mathbf{F}$ along boundaries not on C or $-C'$ will cancel out. Note that the integral on the left hand side of (4) might a priori be very hard to evaluate directly because C might be completely random.

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2. Surfaces

Let us now turn to calculus on a surface. Just like a curve in $\mathbb{R}^n$ is given by a function $\mathbf{r} : [a,b] \to \mathbb{R}^n$ for some interval $[a,b] \subseteq \mathbb{R}$, a surface is given by $\mathbf{r} : \Omega \to \mathbb{R}^n$ for some region Ω in $\mathbb{R}^2$. In this course, we mostly look at surfaces in $\mathbb{R}^3$.

Definition 1. Suppose that Ω is open and let u, v be parameters on Ω. We say that the parametrized surface $\mathbf{r}(u,v)$ is smooth (following the terminology of textbook) if both $\mathbf{r}_u$ and $\mathbf{r}_v$ are continuous and $\mathbf{r}_u \times \mathbf{r}_v$ is everywhere nonzero, or equivalently, the tangent vectors $\mathbf{r}_u$ and $\mathbf{r}_v$ are never colinear. If Ω is closed, then we extend this definition by restricting to the interior of Ω.

Let us consider the surface area of a parametrized surface $\mathbf{r} : \Omega \to \mathbb{R}^3$. For simplicity, let us assume that Ω is rectangular and take a partition P of Ω into sub rectangular regions. Let $R \in P$ and suppose that its sides are given by Δu and Δv. Let (u_R, u_R) be the left bottom corner on R. When R is small, $\mathbf{r}$ is close to being a linear transformation near (u_R, v_R), i.e.,

$$\mathbf{r}(\begin{bmatrix} u - u_R \\ v - v_R \end{bmatrix}) \approx \begin{bmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{bmatrix} \begin{bmatrix} u - u_R \\ v - v_R \end{bmatrix}.$$

Therefore, $\mathbf{r}(R)$ is roughly the rectangle whose sides are

$$\begin{bmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{bmatrix} \begin{bmatrix} \Delta u \\ 0 \end{bmatrix} = \begin{bmatrix} x_u \\ y_u \\ z_u \end{bmatrix} \Delta u = \mathbf{r}_u \Delta u \text{ and } \begin{bmatrix} x_u & x_v \\ y_u & y_v \\ z_u & z_v \end{bmatrix} \begin{bmatrix} 0 \\ \Delta v \end{bmatrix} = \begin{bmatrix} x_v \\ y_v \\ z_v \end{bmatrix} \Delta v = \mathbf{r}_v \Delta v.$$

The area of this rectangle is computed by

$$||\mathbf{r}_u \times \mathbf{r}_v|| \Delta u \Delta v.$$

Therefore, the surface area is computed by

$$\iint_\Omega ||\mathbf{r}_u \times \mathbf{r}_v|| du dv.$$

Example 2. To compute the surface area of a hemisphere of radius 1, we use the parametrization:

$$\mathbf{r}(u,v) = (\sin u \cos v, \sin u \sin v, \cos u)$$

where $u \in [0, \pi/2]$ and $v \in [0, 2\pi]$.

The partial derivatives are:

$$\mathbf{r}_u = (\cos u \cos v, \cos u \sin v, -\sin u)$$
$$\mathbf{r}_v = (-\sin u \sin v, \sin u \cos v, 0)$$

The cross product $\mathbf{r}_u \times \mathbf{r}_v$ is:

$$\mathbf{r}_u \times \mathbf{r}_v = \left(\sin^2 u \cos v, \sin^2 u \sin v, \sin u \cos u\right)$$

The magnitude of the cross product is:

$$||\mathbf{r}_u \times \mathbf{r}_v|| = \sin u$$

The surface area A is then:

$$A = \iint ||\mathbf{r}_u \times \mathbf{r}_v|| \, du \, dv = \int_0^{2\pi} \int_0^{\pi/2} \sin u \, du \, dv = 2\pi$$

You are easily figure out that the surface area of a sphere of radius r is $4\pi r^2$.

Example 3. To compute the surface area of a cone (excluding the bottom) with bottom radius r and height h, we use the parametrization:

$$\mathbf{r}(u,v) = \left(\frac{ru}{h} \cos v, \frac{ru}{h} \sin v, u\right)$$

where $u \in [0, h]$ and $v \in [0, 2\pi]$.The partial derivatives are:

$$\mathbf{r}_u = \left(\frac{r}{h} \cos v, \frac{r}{h} \sin v, 1\right)$$

$$\mathbf{r}_v = \left(-\frac{ru}{h}\sin v, \frac{ru}{h}\cos v, 0\right)$$

The cross product $\mathbf{r}_u \times \mathbf{r}_v$ is:

$$\mathbf{r}_u \times \mathbf{r}_v = \left(-\frac{ru}{h}\cos v, -\frac{ru}{h}\sin v, \frac{r^2 u}{h^2}\right)$$

The magnitude of the cross product is:

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \frac{ru}{h}\sqrt{1+\frac{r^2}{h^2}}$$

The surface area A is:

$$A = \iint \|\mathbf{r}_u \times \mathbf{r}_v\|\, du\, dv = \int_0^{2\pi}\int_0^h \frac{ru}{h}\sqrt{1+\frac{r^2}{h^2}}\, du\, dv$$

Evaluating the integrals:

$$A = \pi r\sqrt{h^2+r^2}$$

Thus, the surface area of the cone (excluding the bottom) is $\pi r\sqrt{h^2+r^2}$. On the other hand, you can verify this formula by flatten the cone to a circular sector (exercise).

Suppose now a surface S is given implicitly as the vanishing set of a function $F(x,y,z)$. Let us find the area of S lying above a region Ω on the xy-plane. Assume that $h(x,y)$ is some function such that $F(x,y,h(x,y)) = 0$. Then we may parametrize the surface as

$$\mathbf{r}(u,v) = u\mathbf{i} + v\mathbf{j} + h(u,v)\mathbf{k}.$$

To compute $\mathbf{r}_u$, we need to know what h_u is. From

$$\frac{\partial}{\partial u}F(u,v,h(u,v)) = (\nabla F)_{(u,v,h(u,v))}\cdot(1,0,h_u) = F_x(u,v,h(u,v)) + h_u F_z(u,v,h(u,v)) = 0$$

we observe that

$$h_u = -\frac{F_x}{F_z}(u,v,h(u,v)).$$

Similarly, we find h_v. This way we find

$$\mathbf{r}_u = \mathbf{i} - \frac{F_x}{F_z}\mathbf{k} \text{ and } \mathbf{r}_v = \mathbf{j} - \frac{F_y}{F_z}\mathbf{k}$$

so that

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \frac{\|\nabla F\|}{|F_z|}.$$

Example 4. Let us ind the area of the surface cut from the paraboloid $x^2+y^2-z=0$ by the plane $z=4$. For the paraboloid $F(x,y,z) = x^2+y^2-z = 0$, the gradient is:

$$\nabla F = (2x, 2y, -1)$$

The magnitude of the gradient is:

$$\|\nabla F\| = \sqrt{4x^2+4y^2+1}$$

Since $F_z = -1$, the surface area formula becomes:

$$A = \iint_\Omega \sqrt{4x^2+4y^2+1}\, dx\, dy$$

The region Ω is a disk of radius 2 in the xy-plane. Switching to polar coordinates:

$$A = \int_0^{2\pi}\int_0^2 r\sqrt{4r^2+1}\, dr\, d\theta$$

Evaluating the inner integral:

$$\int_0^2 r\sqrt{4r^2+1}\, dr = \frac{1}{12}\left(17\sqrt{17}-1\right)$$

Thus, the surface area is:

$$A = \frac{\pi}{6}\left(17\sqrt{17} - 1\right).$$

As a special case, we know that if a surface is given by $z = f(x,y)$ over a region Ω on the xy-plane, then the surface area formula is given by

$$\iint_\Omega \sqrt{f_x^2 + f_y^2 + 1}\,dxdy.$$