LECTURE 15

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Green's Theorem is a fundamental result in vector calculus that relates a line integral around a simple closed curve C to a double integral over the plane region D bounded by C. It is a special case of the more general Stokes' Theorem.

Theorem 1. *Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane, and let D be the region bounded by C. If $M(x,y)$ and $N(x,y)$ have continuous partial derivatives on an open region that contains D, then:*

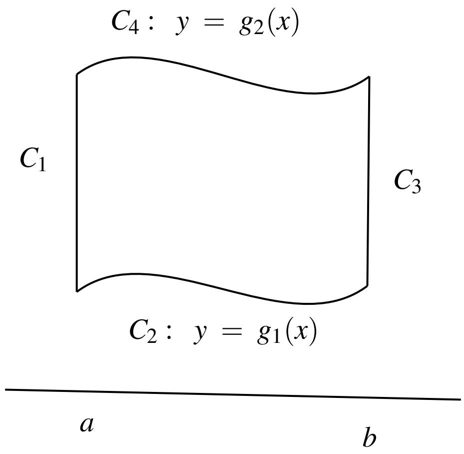
$$\oint_C (M\,dx + N\,dy) = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

The intuition about the above theorem is this: It connects the circulation (line integral) around a closed curve C to the sum of the "microscopic circulations" (curl) inside the region D. It essentially says that the total circulation around the boundary is equal to the sum of all the little swirls inside.

To make the proof accessible, we will consider a simple region D that is both **Type I** and **Type II**. A Type I region is bounded by vertical lines $x = a$ and $x = b$, and by curves $y = g_1(x)$ and $y = g_2(x)$. A Type II region is bounded by horizontal lines $y = c$ and $y = d$, and by curves $x = h_1(y)$ and $x = h_2(y)$. For regions that naturally appear in nature, you can always divide them into a sum of type I and type II regions. Below we only consider Type I regions, we shall prove

$$\oint_C M\,dx = -\iint_D \frac{\partial M}{\partial y} dA$$

For a Type I region, the boundary C (with counterclockwise orientation) consists of four parts:



Now we compute the line integrals.

- On C_2 and C_4, dx is non-zero, while on C_1 and C_3, $dx = 0$.
- Thus, the line integral simplifies to:

$$\oint_C P\,dx = \int_{C_2} M\,dx + \int_{C_4} M\,dx$$

- On C_2, $y = g_1(x)$, and on C_4, $y = g_2(x)$.

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- The integrals become:
$$\oint_C M\,dx = \int_a^b M(x,g_1(x))\,dx - \int_a^b M(x,g_2(x))\,dx$$
(The minus sign arises because C_4 is traversed from $x=b$ to $x=a$.)
- The difference $M(x,g_1(x)) - M(x,g_2(x))$ can be written as:
$$-\int_{g_1(x)}^{g_2(x)} \frac{\partial M}{\partial y}\,dy$$
- Substituting this into the line integral:
$$\oint_C M\,dx = -\int_a^b \left(\int_{g_1(x)}^{g_2(x)} \frac{\partial M}{\partial y}\,dy\right) dx = -\iint_D \frac{\partial M}{\partial y} dA$$

The proofs of the other parts of the theorem are entirely similar.

Theorem 1 can be seen as a shortcut to compute circulation of a vector field. It has another form which you can use to compute flux:
$$\oint_C (Mdy - Ndx) = \iint_D \left(\frac{\partial N}{\partial x} + \frac{\partial M}{\partial y}\right) dA$$
They are two forms of the same theorem. You should have no difficulty converting one to the other.

Example 2. Use Green's Theorem to evaluate the line integral:
$$\oint_C \mathbf{F} \cdot d\mathbf{r},$$
where $\mathbf{F}(x,y) = (x-y)\mathbf{i} + x\mathbf{j}$ and C is the unit circle oriented counterclockwise.

The integrand is:
$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1 - (-1) = 2.$$
Hence we are just integrating the constant function 2 over the unit circle, namely 2 times the area. So the answer is 2π.

Exercise: Please try to verify this answer using our usual way of computing line integrals.