LECTURE 14

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Suppose that C and C' are two oriented curves that both start at a point P and end at a point Q. You may recall that the line integrals of a function $f : \Omega \to \mathbb{R}$, where Ω is a domain containing both C and C', can differ along these curves. A similar phenomenon occurs with line integrals over a vector field, and you can easily construct examples to illustrate this.

Definition 1. Let Ω be an open domain in $\mathbb{R}^n$ and $\mathbf{F}$ be a vector field on Ω. We say that $\mathbf{F}$ is *conservative* if the line integral of $\mathbf{F}$ along an oriented curve depends only on the starting point and the end point. In short, the line integral of $\mathbf{F}$ is *path independent.*

Proposition 2. *A vector field* $\mathbf{F}$ *is conservative if and only if along every oriented closed curve* C*,*

$$\oint_C \mathbf{F} \cdot \mathbf{T} ds = 0. \tag{1}$$

Proof. The "only if" direction is trivial: Let C be an oriented closed loop that starts and end at a point P. Then we can define another curve C' by $[0,0] \to \mathbb{R}^n$ which sends 0 to P. Clearly, the integral of $\mathbf{F}$ along C' is 0, but it has be equal to the integral over C.

For the "if part", suppose that C and C' are two oriented curves that both start at a point P and end at a point Q. Let $-C$ be the curve which is the same as C as a point set but is equipped with the opposite orientation. Then we have

$$\int_{-C} \mathbf{F} \cdot \mathbf{T} ds = -\int_C \mathbf{F} \cdot \mathbf{T} ds.$$

Now note that $C' \cup -C$ is an oriented closed curve from P to itself. Hence by assumption we have

$$\oint_{C' \cup -C} \mathbf{F} \cdot \mathbf{T} ds = \int_{C'} \mathbf{F} \cdot \mathbf{T} ds + \int_{-C} \mathbf{F} \cdot \mathbf{T} ds = \int_{C'} \mathbf{F} \cdot \mathbf{T} ds - \int_C \mathbf{F} \cdot \mathbf{T} ds = 0,$$

which gives us what we want. $\square$

Let us call a vector field $\mathbf{F}$ continuous, or continuously differentiable if it is so as a function $\Omega \to \mathbb{R}^n$. This is equivalent to saying that its components are continous or continuously differentiable.

Theorem 3. *A continuous vector field* $\mathbf{F}$ *is conservative if and only if it admits a differentiable potential function* $f : \Omega \to \mathbb{R}$ *(i.e.,* $\nabla f = \mathbf{F}$*).*

Proof. First, suppose that there exists f such that $\nabla f = \mathbf{F}$. Let C be an oriented curve $\mathbf{r} : [a,b] \to \Omega$. Then we have

$$\int_C \mathbf{F} \cdot \mathbf{T} ds = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_a^b \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_a^b \frac{d}{dt}\big(f(\mathbf{r}(t))\big) dt = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

Then clearly the integral only depends on $\mathbf{r}(a)$ and $\mathbf{r}(b)$, which are the starting point and the end point of C respectively.

Now let us show the converse. Suppose that $\mathbf{F}$ is conservative. Let us choose a point $\mathbf{x}_0 \in \Omega$. For every other $\mathbf{x} \in \Omega$, we choose a curve C which starts from $\mathbf{x}_0$ and ends at $\mathbf{x}$. Again suppose that C is given by $\mathbf{r} : [a,b] \to \Omega$. Then we define

$$f(\mathbf{x}) = \int_C \mathbf{F} \cdot \mathbf{T} ds.$$

This is independent of the choice of C by assumption, and hence the function $f : \Omega \to \mathbb{R}$ is well defined. Then we show that $\nabla f = \mathbf{F}$. Suppose that on $\Omega \subseteq \mathbb{R}^n$ the coordinates are given by $x_1, \cdots, x_n$, and $\mathbf{F}$ is given componentwise by $(F_1, \cdots, F_n)$. Then we have

$$\frac{\partial f}{\partial x_1} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\big(f(x_1 + \varepsilon, x_2, \cdots, x_n) - f(x_1, \cdots, x_n)\big) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{C_\varepsilon} \mathbf{F} \cdot \mathbf{T} ds,$$

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where C_ε is any curve from $(x_1,\cdots,x_n)$ to $(x_1+\varepsilon,x_2,\cdots,x_n)$. We might as well take C to be given by $\mathbf{r}(t)=(x_1+t,x_2,\cdots,x_n)$ for $t\in[0,\varepsilon]$. Now we have

$$\int_{C_\varepsilon}\mathbf{F}\cdot\mathbf{T}ds=\int_0^\varepsilon \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)dt=\int_0^\varepsilon (F_1,\cdots,F_n)\cdot(1,0,\cdots,0)dt=\int_0^\varepsilon F_1dt.$$

By single variable calculus we have

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^\varepsilon F_1dt=F_1$$

as desired. The same argument of course works for every other component F_i. As each F_i is continous by assumption, we have that f is continously differentiable. □

Recall the following caveat: For a multivariable function, if all partial derivatives exist, we may still not have that the function is differentiable. However, if all partial derivatives are continous, then the original function is continuously differentiable.

Corollary 4. *Suppose that* $\mathbf{F}$ *is an orientable continuously differentiable vector field. If* $\mathbf{F}$ *is conservative, then for every* i,j*, we have a symmetry*

$$\frac{\partial F_i}{\partial x_j}=\frac{\partial F_j}{\partial x_i}.$$

This is an immediate consequence of Schwarz theorem: If $f:\Omega\to\mathbb{R}$ is a twice continously differentiable function, then for every i,j

$$\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right)=\frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right).$$

The above corollary gives us a test for conservativity. However, consider the following example.

Example 5. Let $\mathbf{F}$ be the vector field on $\mathbb{R}^2\setminus\{(0,0)\}$ given by

$$\frac{-y}{x^2+y^2}\mathbf{i}+\frac{x}{x^2+y^2}\mathbf{j}.$$

One quickly checks that

$$\frac{\partial}{\partial x}\left(\frac{-y}{x^2+y^2}\right)=\frac{-y^2+x^2}{x^2+y^2}=\frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right).$$

However, let us consider the unit circle C with counterclockwise orientation, parametrized by $t\mapsto(\cos(t),\sin(t))$ for $t\in[0,2\pi]$. Then

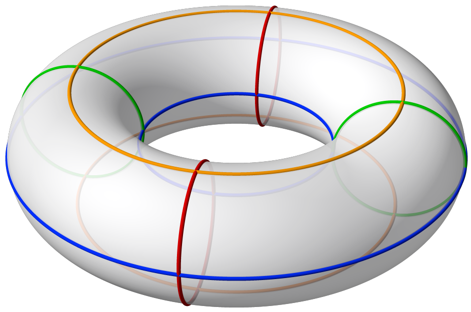
$$\oint_C\mathbf{F}\cdot d\mathbf{r}=\int_0^{2\pi}(-\sin(t)\mathbf{i}+\cos(t)\mathbf{j})\cdot(-\sin(t)\mathbf{i}+\cos(t)\mathbf{j})dt=2\pi\neq 0.$$

Therefore, $\mathbf{F}$ is not conservative.

The above example shows that passing the partial derivative test is a necessary condition for being conservative, but it may not be sufficient. The problem is that, in the region enclosed by the curve C, there is a point where $\mathbf{F}$ is not defined (or more precisely, $\mathbf{F}$ can not be extended to a continuously differentiable function across the point). In this case, the point is $(0,0)$, where the denominators in $\mathbf{F}$ blow up. This problem can be avoided if the region Ω is simply connected.

Definition 6. Let $\Omega\subseteq\mathbb{R}^n$ be a subset. We say that Ω is *simply connected* if it is path connected (i.e., for every $\mathbf{x},\mathbf{x}'\in\Omega$, there exists a curve which starts from $\mathbf{x}$ and ends at $\mathbf{x}'$) and every closed curve (i.e., a loop) can be continuously deformed to a point.

The rigorous definition for "continuously deformed to a point" is this. Suppose that a closed curve is given by $\mathbf{r}:[a,b]\to\Omega$ with $\mathbf{r}(a)=\mathbf{r}(b)$. We say that C can be continuously deformed to a point if there exists a continous function $R:[a,b]\times[0,1]\to\Omega$ such that $R(t,0)=\mathbf{r}(t)$ for every t and $R(t,1)$ sends every t to a single point. For example, $\mathbb{R}^2$ is simply connected. For every curve C, you can always define R by $R(t,s)=s\mathbf{r}(t)$. On the other hand, the punctured plane $\mathbb{R}^2\setminus\{(0,0)\}$ is not simply connected. If you can take any loop which encloses the origin, then you cannot continuously deform the loop to a point without passing through the origin, which is not allowed. On the other hand, by drawing pictures, you can convince yourself that $\mathbb{R}^2\setminus\{(x,0)|x\geq 0\}$ is simply connected.



Detecting whether a surface in 3d is simply connected is actually more fine. For example, a sphere is simply connected, but a torus is not. The colored loops below do not continuously deform to a point.

Now with this extra condition on the domain. We have the following theorem.

Theorem 7. *Let* Ω *be a simply connected open domain of* $\mathbb{R}^n$. *Then a continuously differentiable vector field* $\mathbf{F}$ *on* Ω *is conservative if and only if it satisfies the conclusion of Corollary 4.*