LECTURE 13

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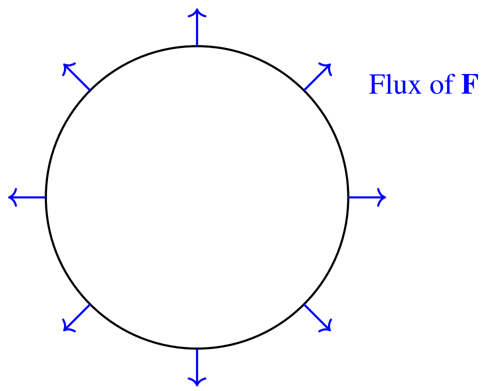
Today we talk about flux of a vector field across a simple closed loop. A closed loop is said to be *simple* if it does not cross itself. For example, a circle is a simple curve, but a figure 8 is not.

Let C be a simple closed curve given by $\mathbf{r} : [a,b] \to \mathbb{R}^2$ with $\mathbf{r}(a) = \mathbf{r}(b)$. Then you can make sense of the interior and the exterior of the region enclosed by the curve. Let $\mathbf{n}$ be the unit normal vector field on C pointing outwwards. That is, at each $\mathbf{x} \in C$, $\mathbf{n}(\mathbf{x})$ is the unit vector which is orthogonal to the tangent line of C at $\mathbf{x}$, and is pointing into the exterior.

Suppose now that $\mathbf{F}$ is a vector field defined on an open Ω which contains C. We define

$$\text{Flux} = \oint_C \mathbf{F} \cdot \mathbf{n} ds,$$

where as usual s is the unit speed parameter. If you imagine $\mathbf{F}$ as the velocity vector field of some liquid flowing in Ω, then the flux measures the rate at which the liquid is flowing outside the region enclosed by C. Here is a picture.



The **cross product** of two vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ in $\mathbb{R}^3$ is a vector $\mathbf{a} \times \mathbf{b}$ that is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. Its magnitude is given by $\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\| \sin\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. The direction of $\mathbf{a} \times \mathbf{b}$ is determined by the **left-hand rule**: if you point the fingers of your left hand in the direction of $\mathbf{a}$ and curl them toward $\mathbf{b}$, your thumb will point in the direction of $\mathbf{a} \times \mathbf{b}$. The cross product can also be computed using the determinant of a 3×3 matrix:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2 b_3 - a_3 b_2)\mathbf{i} - (a_1 b_3 - a_3 b_1)\mathbf{j} + (a_1 b_2 - a_2 b_1)\mathbf{k},$$

where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the standard unit vectors in $\mathbb{R}^3$. Geometrically, the cross product represents the area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$, and it is antisymmetric, meaning $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$.

Suppose that you take the counterclockwise orientation of the curve C, and C is given by $\mathbf{r} : [a,b] \to \mathbb{R}^2$. Then for each t, the unit tangent vector at the point $\mathbf{r}(t)$ is $\mathbf{r}'(t)$. Let $\mathbf{T}$ be the normalized tangent vector $\mathbf{r}'(t)/\|\mathbf{r}'(t)\|$. Then the normal vector $\mathbf{n}$ pointing outwards is given by $\mathbf{T} \times \mathbf{k}$.

Suppose now that $\mathbf{F}$ is given componentwise by $M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$. Let us use the unit speed parametrization s. Then

$$\mathbf{T} \times \mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ dx/ds & dy/ds & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{dy}{ds}\mathbf{i} - \frac{dx}{ds}\mathbf{j}.$$

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Now
$$\oint_C (M\mathbf{i}+N\mathbf{j})\cdot(\frac{dy}{ds}\mathbf{i}-\frac{dx}{ds}\mathbf{j})ds=\oint_C (M\frac{dy}{ds}-N\frac{dx}{ds})ds=\oint_C Mdy-Ndx.$$

Please be cautious that "cancelling out ds" is not a mathematically rigorous operation. It is just a heuristic device for the change of variable formula.

The expression $Mdy-Ndx$ is an example of a differential 1-form. Basically, an 1-form is something which makes sense to be integrated over a curve. For example, suppose that $\mathbf{F}=-y\mathbf{i}+x\mathbf{j}$ and the curve C is the curve $r=2+2\cos(\theta)$. The flux of the vector field $\mathbf{F}=(-y,x)$ along C is computed as follows:

$$\text{Flux}=\oint_C \mathbf{F}\cdot\mathbf{n}\,ds=\oint_C(-y)\,dy-x\,dx.$$

Substituting the parametrization $x=(2+2\cos(\theta))\cos(\theta)$ and $y=(2+2\cos(\theta))\sin(\theta)$, we compute dx and dy:

$$\frac{dx}{d\theta}=-2\sin(\theta)(1+2\cos(\theta)),$$
$$\frac{dy}{d\theta}=2\cos(\theta)+2\cos^2(\theta)-2\sin^2(\theta).$$

Substituting into the flux integral:
$$\text{Flux}=\int_0^{2\pi}\left[-(2+2\cos(\theta))\sin(\theta)\cdot\frac{dy}{d\theta}-(2+2\cos(\theta))\cos(\theta)\cdot\frac{dx}{d\theta}\right]d\theta.$$

After simplification, the integrand becomes:
$$\text{Flux}=\int_0^{2\pi}12\cos^2(\theta)\,d\theta.$$

Using the identity $\cos^2(\theta)=(1+\cos(2\theta))/2$, we evaluate the integral:
$$\text{Flux}=6\int_0^{2\pi}(1+\cos(2\theta))\,d\theta=6\cdot 2\pi=12\pi.$$