LECTURE 12

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Let me switch to using $\mathbf{r}$ *instead of* γ *like your textbook because the boldface* γ *in Latex does not look very different.*

In general, line integral of a function depends on the path, not just on the starting point and the ending point. For example, consider the function $f(x,y,z) = x - 3y^2 + z$. Let C_1 be the straight line from $(0,0,0)$ to $(1,1,1)$, C_2 be that from $(0,0,0)$ to $(1,1,0)$, and C_3 be that $(1,1,0)$ to $(1,1,1)$. Then we can check that

$$\int_{C_1} f = 0, \text{ but } \int_{C_2 \cup C_3} f = \int_{C_2} f + \int_{C_3} f = -\frac{\sqrt{2}}{2} - \frac{3}{2}.$$

Next, we move on to talk about integration of a vector field along a curve. First, let $\Omega \subseteq \mathbb{R}^n$ be a region (we normally consider $n = 2,3$). Then a vector field $\mathbf{F}$ on Ω is what associates to each point $\mathbf{x} \in \Omega$ a vector in $\mathbb{R}^n$.[1] Therefore, in practice $\mathbf{F}$ is given by a function $\Omega \to \mathbb{R}^n$. In $\mathbb{R}^2$, we often write $\mathbf{F}$ componentwise as

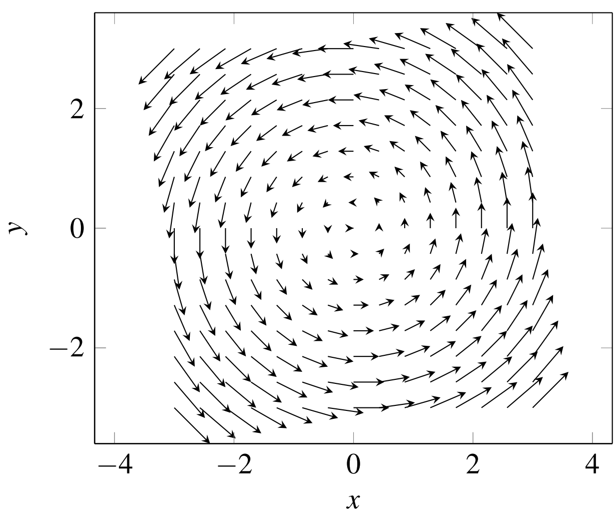
$$\mathbf{F}(x,y) = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$$

for suitable functions $M, N : \Omega \to \mathbb{R}$. Similarly, in $\mathbb{R}^3$, we will add a component

$$\mathbf{F}(x,y,z) = M(x,y,z)\mathbf{i} + N(x,y,z)\mathbf{j} + L(x,y,z)\mathbf{k}.$$

Below is an example of a vector field.

Vector Field $\mathbf{F}(x,y) = -y\mathbf{i} + x\mathbf{j}$



When observing the vector field, you can see that it appears to rotate around the origin. This rotational behavior arises because, at each point $(x,y) \in \mathbb{R}^2$, the vector $-y\mathbf{i} + x\mathbf{j}$ is orthogonal to the position vector $x\mathbf{i} + y\mathbf{j}$. Additionally, the length of the vector $\mathbf{F}(x,y)$ increases with distance from the origin. Indeed, the length of $\mathbf{F}(x,y)$ is given by $\sqrt{x^2 + y^2}$, meaning that vectors farther from the origin are longer.

When Ω is open, and $f : \Omega \to \mathbb{R}$ is a function, we can always associate a gradient vector field ∇f. For general $\mathbb{R}^n$ with coordinates $x_1, \cdots, x_n$, the formula is given by

$$\nabla f = (\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}).$$

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[1]Technically, this latter $\mathbb{R}^n$ should be thought of as the tangent space of the ambient $\mathbb{R}^n$ at $\mathbf{x}$.

For example, suppose that $f(x,y,z) = x^2y + z$, then

$$\nabla f = 2xy\mathbf{i} + x^2\mathbf{j} + \mathbf{k}.$$

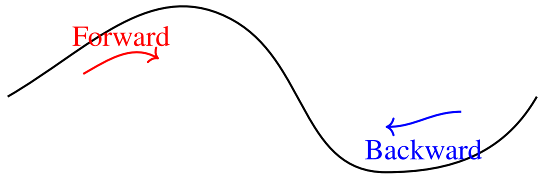
Suppose now that $\mathbf{r} : [a,b] \to \mathbb{R}^n$ is a curve. We define the integration of $\mathbf{F}$ along C by

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt. \tag{1}$$

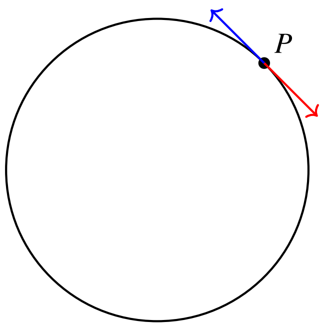
Note that the dot product $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)$ is a function in t, so that the right hand side is just the usual integral of a function.

Now you are rightfully worried about to what extent this integral is well defined, i.e., independent of the choice of the parametrization. It turns out that it is well defined once you choose an orientation for C. Here is a picture.

Curve with Two Orientations



A choice of an orientation gives you a consistent choice of a *unit* tangent vector $\mathbf{T}$ at each point on the curve. For example, you can parametrize a unit circle by $t \mapsto (\cos(t), \sin(t))$. Then when you take the velocity vector, you get $(-\sin(t), \cos(t))$, pointing in the counterclockwise direction (the blue arrow below). If you change t by $-t$, then you are still parametrizing the same circle, but now your tangent vector becomes $(\sin(t), \cos(t))$, which is pointing in the clockwise direction (the red arrow below).



Given a parametrization $\mathbf{r} : [a,b] \to \mathbb{R}^n$ of a curve C, you can easily figure out the orientation by looking at how the point moves as t increases. You can always flip the orientation by a sign: Say you define $\bar{\mathbf{r}} : [-b,-a] \to \mathbb{R}^n$ by $\bar{\mathbf{r}}(t) = \mathbf{r}(-t)$. Then $\bar{\mathbf{r}}$ will have the opposite orientation.

Given $\mathbf{r}$, we define the unit speed reparametrization by introducing a parameter s defined by

$$s(t) = \int_a^{a+t} \|\mathbf{r}'(u)\| du.$$

Then we have

$$\frac{ds}{dt} = \|\mathbf{r}'(t)\|.$$

Let us ignore the extreme case that the curve C is just a point. Then $t \mapsto s(t)$ defines a bijection between $[a,b]$ and $[0, \text{length}(C)]$. Let $\widetilde{r}(s)$ be the function such that $\widetilde{\mathbf{r}}(s(t)) = \mathbf{r}(t)$. Then by the chain rule we have

$$\frac{d\widetilde{\mathbf{r}}}{ds} \cdot \frac{ds}{dt} = \frac{d\mathbf{r}}{dt} \Longrightarrow \widetilde{\mathbf{r}}'(s) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \Rightarrow \|\widetilde{\mathbf{r}}'(s)\| = 1.$$

The unit tangent vector $\mathbf{T}$ is nothing but $\widetilde{\mathbf{r}}'(s)$. The parametrization $\widetilde{\mathbf{r}}(s)$ depends only on the orientation of C chosen by $r(t)$ but other than that, it does not depend on $\mathbf{r}(t)$.

To sum up, given an oriented curve C, we have a canonically defined unit speed parametrization with parameter s. Then we can define the integral of $\mathbf{F}$ along C as

$$\int_C (\mathbf{F} \cdot \mathbf{T}) ds.$$

This agrees with (1).

Line integrals of vector fields have important physical interpretations. The first is the **work done by a force field F** along a curve C. If $\mathbf{F}$ represents a force acting on an object, the line integral $\int_C \mathbf{F} \cdot \mathbf{T} ds$ computes the work done by the force as the object moves along C. The second interpretation is the **circulation of a flow** in fluid dynamics. If $\mathbf{F}$ represents the velocity field of a fluid, the line integral $\oint_C \mathbf{F} \cdot \mathbf{T} ds$ measures the circulation of the fluid around the closed curve C. Circulation quantifies the tendency of the fluid to rotate around C, and it is particularly useful in studying vortices and rotational flows. For example, suppose that we consider the figure in page 1 and we compute the circulation around the circle of radius r with counterclockwise orientation. Then we have

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (-\sin(\theta)\mathbf{i} + \cos(\theta)\mathbf{j}) \cdot (-r\sin(\theta)\mathbf{i} + r\cos(\theta)\mathbf{j}) d\theta = \int_0^{2\pi} r d\theta = 2\pi r$$

which is precise the length of the circle C.