LECTURE 11

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Let us wrap up the change-of-variable formula by looking at an example in the textbook:

$$\int_1^2 \int_{1/y}^{y} \sqrt{\frac{y}{x}} e^{\sqrt{xy}} dxdy.$$

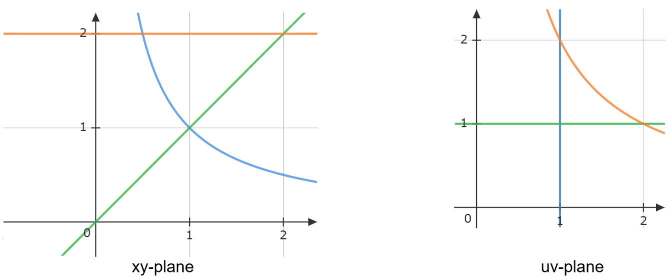
We do this by the change of variables

$$u = \sqrt{xy} \text{ and } v = \sqrt{\frac{y}{x}}.$$

Then by solving equations we obtain that

$$x = \frac{u}{v} \text{ and } y = uv.$$

The change of regions of integration is graphed below:



Let the enclosed region on the left be Ω and the one on the right be Ω'. The function $g : \Omega' \to \Omega$ is given by $g(u,v) = (u/v, uv)$. We quickly compute that $|\det(J_g)| = 2u/v$, so that by the change-of-variable formula we get

$$\int_1^2 \int_{1/y}^{y} \sqrt{\frac{y}{x}} e^{\sqrt{xy}} dxdy = \int_1^2 \int_1^{2/v} ve^u \frac{2u}{v} dudv = 2e(e-2).$$

(Added after class) Let me remark that in setting up the new integral, you do not figure out the new limits of integration by a word-for-word translation of the old one. For example, $y = 1$ is a lower limit in the old integral. However, its literal translation $uv = 1$ does not show up in the limits of the new integral. This is in fact a coincidence. If we change $y = x$ to $y = x/2$ in the original integral, then you see the translation of $y = 1$:

This time, the change of variable formula is set up as

$$\int_1^2 \int_{1/y}^{2y} \sqrt{\frac{y}{x}} e^{\sqrt{xy}} dxdy = \int_1^2 \int_1^{2/v} ve^u \frac{2u}{v} dudv + \int_{1/2}^1 \int_{1/v}^{2/v} ve^u \frac{2u}{v} dudv.$$

In particular, you need to break Ω' into two parts, but then you see that the line with $y = uv = 1$ indeed shows up.

For the time remaining we give a short introduction to line integrals.

Let C be a (close and bounded) curve in $\mathbb{R}^n$. In our class, this shall mean a subset of $\mathbb{R}^n$ which is the image of some continuously differentiable function $\gamma : [a,b] \to \mathbb{R}^n$. In practice, we usually only

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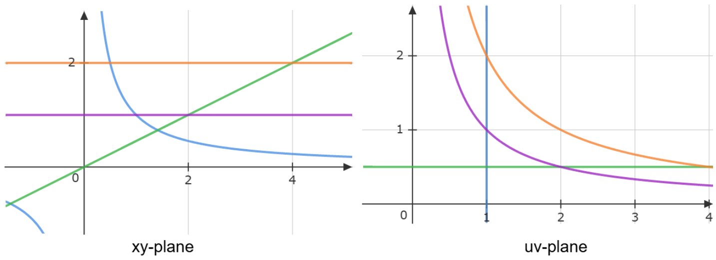

consider $n = 2, 3$ and we allow the curve γ to be piecewise continuously differentiable. The function γ is a parametrization of the curve C, but the same curve may have many different parametrizations. For example, the upper semi-circle can be viewed as the image of

$$\gamma : [0, \pi] \to \mathbb{R}^2 \quad \gamma(\theta) = (\cos(\theta), \sin(\theta))$$

or

$$\gamma : [-1, 1] \to \mathbb{R}^2 \quad \gamma(t) = (t, \sqrt{1-t^2}).$$

As usual, given a function $f : C \to \mathbb{R}$, we say that f is integrable if the limit

$$\lim_{\|P\| \to 0} L(P, f)$$

exists. Here P is a partition of the interval $[a, b]$, and $L(P, f)$ is the Riemann sum given by (say $P = \{a = t_0, t_1, \cdots, t_n = b\}$)

$$L(P, f) = \sum_{i=0}^{n-1} f(\gamma(t_i)) \cdot \|\gamma(t_i) - \gamma(t_{i+1})\|.$$

The term $\|\gamma(t_i) - \gamma(t_{i+1})\|$ is of course the distance between these two points in $\mathbb{R}^n$. Note that when $f = 1$, the integral $\int_C 1$ is nothing but the length of the curve.

When f is continuous, we can compute the integral $\int_C f$ by

$$\int_C f = \int_a^b f(\gamma(t)) \|\gamma'(t)\| dt.$$

Note that $\gamma'(t)$ is the velocity vector of the parametrization at t. By the change of variable formula, you can verify that the integral $\int_C f$ does not depend on the choice of the parametrization.