LECTURE 9

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In order to introduce the general change of variable formula for integrals, let us do a brief recap on linear transformations.

Definition 1. A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is called a linear transformation if it respects addition and scalar multiplication, i.e.,

(i) $\forall v, w \in \mathbb{R}^n$, $T(v+w) = T(v) + T(w)$.
(ii) $\forall c \in \mathbb{R}, v \in \mathbb{R}^n$, $T(cv) = cT(v)$.

Example 2. On $\mathbb{R}^2$, rotation-by-θ is a linear transformation for every angle θ. Let us call it T. It is clear that T commutes with scaling. To see that it commutes with addition, it suffices to observe that the formation of the parallelogram with vertices $0, v, w, v+w$ commutes with T.

Definition 3. Let $e_1, e_2, \cdots, e_n$ be the standard basis of $\mathbb{R}^n$, i.e., e_i is the vector with an 1 on the ith entry and 0 everywhere else. The matrix A associated to a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is given by

$$A = \begin{bmatrix} \vdots & \vdots & \cdots & \vdots \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ \vdots & \vdots & \cdots & \vdots \end{bmatrix}$$

Note that each $T(e_i)$ has length m, and hence A is a $m \times n$ matrix.

What is amazing about having this matrix is that, in order to figure out what it is, you only need to consider how T acts on the e_i's, but once you have it, it works for an arbitrary vector. Namely, for any $v \in \mathbb{R}^n$, $T(v) = Av$. To see this, suppose that v has coordinates $\begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}^T$, which means that $v = \sum v_i e_i$. Then we have

$$T(v) = T(\sum_{i=1}^n v_i e_i) = \sum_{i=1}^n v_i T(e_i) = \sum_{i=1}^n v_i A(e_i) = A(\sum_{i=1}^n v_i e_i) = Av.$$

Here, the second equation uses that T is a linear transformation, and the second-to-last equation uses that multiplication by a matrix is also always a linear transformation.

Example 4. Consider the rotation-by-θ transformation T in Example 2. It is easy to figure out that

$$T(e_1) = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} \text{ and } T(e_2) = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}.$$

Therefore, to obtain the matrix A associated to T, we just need to put these columns together:

$$A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

For any other

$$v = \begin{bmatrix} a \\ b \end{bmatrix} \text{ we have } T(v) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a\cos(\theta) - b\sin(\theta) \\ a\sin(\theta) + b\cos(\theta) \end{bmatrix}.$$

Next, let us recall how to compute determinants. For a 2×2 matrix, we have a simple formula

$$\det(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = ad - bc.$$

For a 3×3 matrix, we need the notion of minors. In general, for a $n \times n$ matrix A, the (i, j)th minor $A_{i,j}$ is the matrix obtained by deleting the ith row and the jth column. Suppose now that the (i, j)th entry

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of A is a_{ij}. Then we can compute $\det(A)$ by expanding any row or column. For example, if we use the first row, then the formula is

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(A_{1j}).$$

If we use the second column, then the formula is

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+2} a_{i2} \det(A_{i2}).$$

Fact: For any region $\Omega \subseteq \mathbb{R}^n$ for which the volume is defined, and for any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$, $|\det(T)|$ is the scaling factor for the change of volume, i.e.,

$$\mathrm{vol}(T(\Omega)) = |\det(T)|\mathrm{vol}(\Omega).$$

Let us do a reality check: When T is the rotation-by-θ transformation, $\det(T) = 1$. This implies by the above that rotation-by-θ does not change the area of a region, which is clear. If you think about the fact that $\det(T) = 1$, you realize that it is nothing but the Pythagorean theorem. Therefore, from the POV of linear algebra, Pythagorean theorem is simply saying that area is invariant under rotation!

I also give a preview of the general change-of-coordinate formula for integrals in terms of Jacobians. I will go through this topic more thoroughly on Friday.