LECTURE 8

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In the 2d case, you can use either the standard xy-coordinates or the polar coordinates. When you do 3d integrals, you may also integrate along z as usual and obtain a function in (x,y), which you can then integrate using polar coordinates. The resulting coordinate system, i.e., (r,θ,z) is called cylindrical coordinates. Since nothing is going on for the z-coordinate, we have:

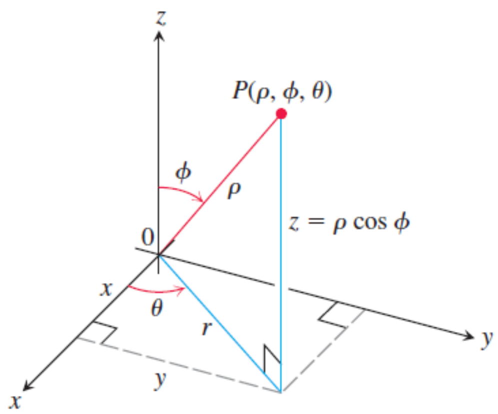
Theorem 1. *The change-of-coordinate formula for integrals in terms of cylindrical coordiantes with respect to the rectangular coordinates is given by*

$$dxdydz = rdzdrd\theta. \tag{1}$$

Example 2. Suppose we want to integrate the function $f(x,y,z)$ over the region Ω above the cylinder $z = \sqrt{x^2+y^2}$ and below the plane $z = 4$. Then in cylindrical coordinates the integral is set up as

$$\int_0^{2\pi}\int_0^2\int_r^4 f(r\cos\theta, r\sin\theta, z) rdzdrd\theta.$$

Besides cylindrical coordinates, we can also use spherical coordinates for $\mathbb{R}^3$. Here is the picture:



In a sense, you can think of spherical coordinates as “cylindrical coordiantes” squared. Namely, we first convert the xy-plane to $r\theta$-plane. Then, for each fixed θ, we convert the rz-plane to $\rho\phi$-plane. To figure out the change-of-coordinate formula, let us first convert (ρ,ϕ) to (z,r) when θ is fixed:

$$r = \rho\sin(\phi), z = \rho\cos(\phi).$$

Then we convert (r,θ) to (x,y), which is given by the usual

$$x = r\cos(\theta), y = r\sin(\theta).$$

The relation of (ρ,ϕ) to (z,r) is the same as the one of (r,θ) to (x,y). To sum up, the change-of-coordinate formula is given by

$$x = \rho\sin(\phi)\cos(\theta), y = \rho\sin(\phi)\sin(\theta), z = \rho\cos(\phi).$$

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Theorem 3. *The change-of-coordinate formula for integrals in terms of spherical coordiantes with respect to the rectangular coordinates is given by*

$$dxdydz = \rho^2 \sin(\phi) d\rho d\theta d\phi. \tag{2}$$

Instead of justifying this particular formula, I will wait for the general theorem of change-of-coordiante formula for integrals. For now, you can just accept it as a blackbox. As a reality check, we can start by computing the volume of the 3-dimensional ball $\mathbb{B}^3$ of radius a. Note that in terms of spherical coordinates, the ball $\mathbb{B}^3$ is given by

$$\{(\rho,\theta,\phi) \in \mathbb{R}^3 \mid \rho \in [0,a], \theta \in [0,2\pi], \phi \in [0,\pi]\}.$$

In other words, the ball $\mathbb{B}^3$ becomes a rectangular box $[0,a] \times [0,2\pi] \times [0,\pi]$. Let us call the latter R. In class, I got myself confused for a moment about why the bounds for ϕ goes from 0 to π as opposed to 2π. Here is the reason: The map $R \to \mathbb{B}^3$ defined by $x \mapsto \rho\sin(\phi)\cos(\theta), y \mapsto \rho\sin(\phi)\sin(\theta), z \mapsto \rho\cos(\phi)$ is a bijection. But if you let ϕ to go from 0 to 2π in the definition of R, then this map will be a 2-to-1 map for most points on $\mathbb{B}^3$. Indeed, if (ρ,θ,ϕ) is such a point, and $\theta < \pi$, then (ρ,θ,ϕ) and $(\rho,\theta+\pi,2\pi-\phi)$ get mapped to the same point.

Now, the volume of $\mathbb{B}^3$ is computed by

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^2 \sin(\phi) dr d\phi d\theta = \frac{4}{3}\pi a^3.$$

This agrees with the formula which you should have known already.

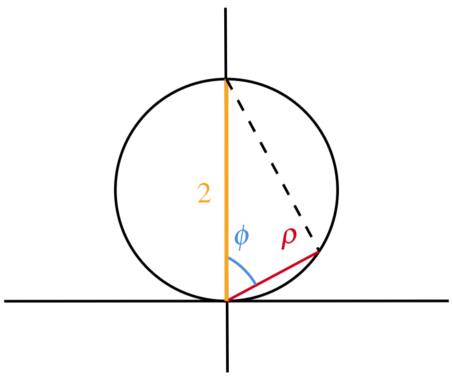
Example 4. What is the surface cut out by the equation $x^2+y^2+(z-1)^2 = 1$ in terms of the spherical coodinates? Well, the most naive thing you can do is to do a brute force computation: Plug in $x \mapsto \rho\sin(\phi)\cos(\theta), y \mapsto \rho\sin(\phi)\sin(\theta), z \mapsto \rho\cos(\phi)$ to get

$$\big(\rho\sin(\phi)\cos(\theta)\big)^2 + \big(\rho\sin(\phi)\sin(\theta)\big)^2 + \big(\rho\cos(\phi)\big)^2 = 1,$$

which simplifies to

$$\rho = 2\cos(\phi). \tag{3}$$

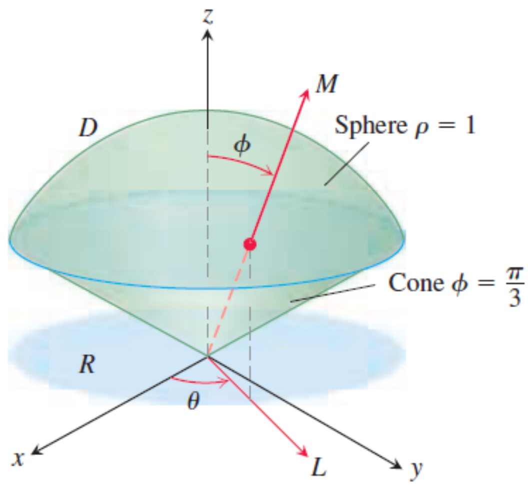
On the other hand, you can quickly figure out the answer by some geometry. You know what $x^2+y^2+z^2 = 1$ looks like—it is just the unit sphere centered at the origin. Now, if you change z to $z-1$, then this amounts moving the picture upwards by 1 unit. You end up getting a unit sphere centered at $(0,0,1)$ instead, so that the sphere is tangent to the xy-plane. If you draw a picture, then you see there is no constraint on θ, so that θ does not appear in the equation, and that ρ and ϕ are related by (3) because of the following picture:



Example 5. Let us compute the volume of the ice cream cone Ω as below:

One quickly figures out that in terms of spherical coordiantes, Ω is given by the rectangular box

$$\{(\rho,\theta,\phi) \in \mathbb{R}^3 \mid \rho \in [0,1], \theta \in [0,2\pi], \phi \in [0,\pi/3]\}.$$



Therefore, we set up the integral as

$$\int_0^{2\pi}\int_0^{\pi/3}\int_0^1 \rho\sin(\phi)drd\theta d\phi,$$

which one quickly evaluates to be $\pi/3$.

Now let us check our answer using cylindrical coordinates. The projection of Ω to the xy-plane is given by the intersection circle of the sphere with the cone $\phi = \pi/3$. The circle has radius $\sqrt{3}/2$, so the circle is given by $x^2 + y^2 = 3/4$. Now fix any (x, y) within the circle, the z runs from the intersection point of $\phi = \pi/3$ with the vertical line through (x, y) to that of the sphere with the same line. Therefore, the region Ω in terms of the cylindrical coordinates is given by

$$\{(r,\theta,z) \in \mathbb{R}^3 \mid r \in [0, \frac{\sqrt{3}}{2}], \theta \in [0, 2\pi], z \in [\frac{r}{\sqrt{3}}, \sqrt{1-r^2}]\},$$

so that the integral is

$$\int_0^{2\pi}\int_0^{\sqrt{3}/2}\int_{r/\sqrt{3}}^{\sqrt{1-r^2}} rdzdrd\theta = 2\pi\int_0^{\sqrt{3}/2} r\big(\sqrt{1-r^2} - \frac{r}{\sqrt{3}}\big)dr.$$

After some tedious computation, we find that the integral is also $\pi/3$.

Hopefully this example gives you some idea on when to use spherical coordinates and when to use cylindrical coordinates.