LECTURE 7

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Let us look at a fun application of polar coordinates. Suppose that we want to evaluate the integral

$$\int_0^\infty e^{-x^2}dx.$$

Then it is hard to do directly, because the function e^{-x^2} does not have an elementary antiderivative. In order to use polar coordinates, we first need to convert the problem to one about double integrals. The trick is to consider the product

$$\Big(\int_0^\infty e^{-x^2}dx\Big)\Big(\int_0^\infty e^{-y^2}dy\Big) = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}dxdy.$$

The region of integration is the first quadrant. Now let us compute this integral using polar coordinates:

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}dxdy = \int_0^{\pi/2}\int_0^\infty e^{-r^2}\cdot rdrd\theta.$$

We know how to evaluate the integral above, and the answer is $\pi/4$. However, note that this is the square of the original integral we want to evaluate, which is hence $\sqrt{\pi}/2$.

Now let us move on triple integrals. The theory of triple integrals is the natural extension of that of double integrals. First, suppose that Ω is a bounded rectangular box (or a cuboid) in $\mathbb{R}^3$ and $f : \Omega \to \mathbb{R}$ is a function. We consider partitions P which divide Ω into smaller rectangular boxes, and then define

$$L(P,f) = \sum_{B\in P} f(\mathbf{x}_B)\mathrm{vol}(B),$$

where $\mathbf{x}_B$ is a chosen corner of B. In fact, this choice does not matter, but to fix ideas we can always choose the corner of B where the coordinates x,y,z all reach the minimum. We define $\|P\|$ to be the volume of the biggest box (or boxes) in P. Then we say that f is integrable if the limit

$$\lim_{\|P\|\to 0} L(P,f)$$

exists, in which case the limit is defined to be the integral $\int_\Omega f$.

Now, suppose that Ω is a general bounded region in $\mathbb{R}^3$. Then to define integrals we choose a rectangular box Ω' big enough to contain Ω, and we define $\widetilde{f} : \Omega' \to \mathbb{R}$ to be the function which is f on Ω and is 0 elsewhere. Then we say that f is integrable over Ω if $\widetilde{f}$ is integrable over Ω', in which case we define

$$\int_\Omega f = \int_{\Omega'} \widetilde{f}.$$

Clearly, the choice of Ω' does not matter, as long as it contains Ω, because $\widetilde{f}$ is 0 outside Ω.

As you can imagine, there is a 3d version of Fubini theorem, which we use in pracice to compute the integrals. Suppose that Ω is a closed and bounded region in $\mathbb{R}^3$. In order to set up the bounds for the triple integral. Consider the projection of Ω to the xy-plane and let R be the image. Then, for each point $(x,y)\in R$, we find the functions $g_1(x,y)$ and $g_2(x,y)$ in terms of x,y such that

$$\Omega = \{(x,y,z)\in\mathbb{R}^3 \mid (x,y)\in R, g_1(x,y)\le z\le g_2(x,y)\}.$$

Then we have

$$\int_\Omega f = \iint_R \Big(\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)dz\Big)dxdy$$

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