LECTURE 6

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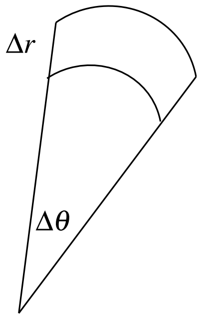
Recall that in single variable calculus, you need a correction factor if you change variables in your integrals. A similar thing happens in higher dimensions. Now we give a heuristic on the formula

$$dxdy = rdrd\theta.$$

Consider the following picture:

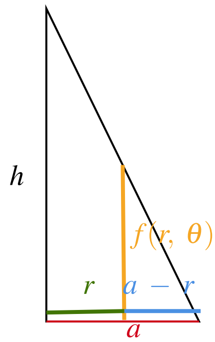


For a sector with radius r and angle θ, its area is $(1/2)r^2\theta$. If you cannot remember the factor, keep in mind that when $\theta = 2\pi$, the area is π. Therefore, when r is replaced by $r+\Delta r$ and θ is replaced by $\theta+\Delta\theta$. The growth of the area is

$$\frac{1}{2}(r+\Delta r)^2\Delta\theta - \frac{1}{2}r^2\theta = r\Delta r\Delta\theta.$$

You can think of an infinitesimally small quantity as being something that is not 0, but whose squares (or products) are 0. In the above "equality", I threw away the higher order (>1) degree terms. (Side note: I made a mistake in class in that I used $\theta+\Delta\theta$ instead of just $\Delta\theta$ in the left hand side of the above equation. The reason that using $\theta+\Delta\theta$ is incorrect is that we are really using small chunks as the above to tile the region of integration.)

Example 1. Let us compute the volume of a cone. Suppose that the height is h and the bottom circle has radius r. Then it is well known that its volume is one third of the corresponding cylinder with the same radius and height. First, we need to figure out the formula for the height given radius r. Note that the height does not depend on θ, which is clear from geometry.



With a little trigonometry, we know that

$$\frac{a-r}{a} = \frac{f(r,\theta)}{h} \Rightarrow f(r,\theta) = h\frac{a-r}{a}.$$

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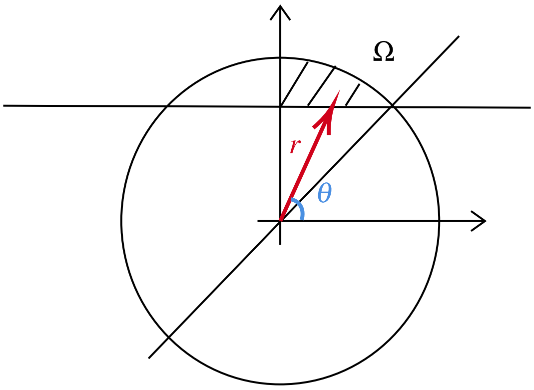
Therefore, to compute the volume, we set up the integral as

$$\int_0^{2\pi}\int_0^a \left(h\frac{a-r}{a}\right) r dr d\theta = \frac{1}{3}\pi a^2 h.$$

Example 2. Let us compute the area of the region below. Suppose the that radius of the circle is 2:



Again, a little trignometry tells us that

$$\frac{\sqrt{2}}{r} = \sin(\theta) \Rightarrow r = \sqrt{2}\csc(\theta).$$

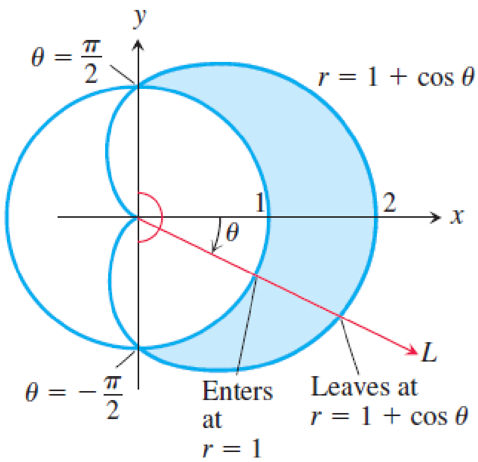
Hence we set up the integral as

$$\int_{\pi/4}^{\pi/2}\int_{\sqrt{2}\csc(\theta)}^{2} r dr d\theta = \frac{1}{2}\int_{\pi/4}^{\pi/2} 2\theta - (\sqrt{2}\csc(\theta))^2 d\theta = \frac{\pi}{2} - 1.$$

Recall that

$$\frac{d}{d\theta}\cot(\theta) = -\csc(\theta)^2.$$

Example 3. Let us compute the following area:



Clearly, the integral is

$$\int_{-\pi/2}^{\pi/2}\int_1^{1+\cos(\theta)} r dr d\theta = \frac{1}{2}\int_{-\pi/2}^{\pi/2}\left((1+\cos(\theta))^2 - 1\right) d\theta = 2 + \frac{\pi}{4}.$$

The other examples we talked about are all taken from your textbook §15.4 like the above. Please read through them carefully.