# LECTURE 5

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Let $\Omega$ be a region on $\mathbb{R}^2$. The area of $\Omega$ is defined as the integral

$$\int_\Omega 1$$

where the 1 as the integrand should be interpreted as the constant function $\Omega \to \mathbb{R}$ which sends every point $x \in \Omega$ to 1.

Suppose we want to find the area $\Omega$ bounded by the curves $y = x^2$ and $y = x+2$. The two curves intersect at points $(-1,1)$ and $(2,4)$. If we use the vertical sections, then the double integral is set up as

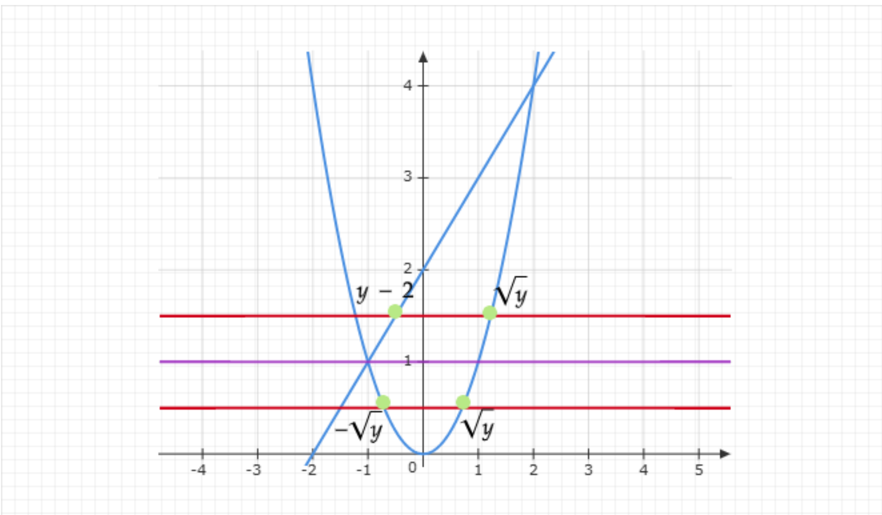
$$\int_{-1}^{2}\int_{x^2}^{x+2} 1 dydx = \int_{-1}^{2} (x-2) - x^2 dx = \frac{9}{2}.$$

Note that if you simply use the fact that the integral computes the area below the graph of a function, without thinking about the double integrals, then you would compute the area below $y = x+2$ and substract the area below $y = x^2$, so that the expression you will set up to compute the area of $\Omega$ is the difference

$$\int_{-1}^{2} (x-2)dx - \int_{-1}^{2} x^2 dx.$$

So considering the double integrals just gives another way of interpreting the same thing.

However, the advantage of comsidering the double integral is that you have the freedom to compute the integral in the other order. For example, now let us compute $\Omega$ by first integrate along $x$ and then along $y$. Then we make use of horizontal sections instead of vertical sections, and we need to divide $\Omega$ into two subregions because their bounds are given by different expressions.




For the area of the subregion below the ling $y = 1$, the double integral is set up as

$$\int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} 1dxdy = \int_0^1 2\sqrt{y}dy = \frac{4}{3}.$$

For the area of the subregion below the ling $y = 1$, the double integral is set up as

$$\int_1^4 \int_{y-2}^{\sqrt{y}} 1dxdy = \int_1^4 \sqrt{y} - y + 2dy = \frac{19}{6}.$$

You may check that they indeed add up to $9/2$.

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Here is a definition from the textbook, which is a useful terminology to know: Given an integrable function $f : \Omega \to \mathbb{R}$, the average value of $f$ over $\Omega$ is defined to be

$$\frac{1}{\mathrm{Area}(\Omega)} \int_\Omega f.$$

In one variable calculus, it is a useful technique to change the variable. For example, suppose that $g : [c,d] \to [a,b]$ is an invertible differentiable function. Then for an integrable function $f : [a,b] \to \mathbb{R}$, we may set $x = g(t)$ and obtain the formula

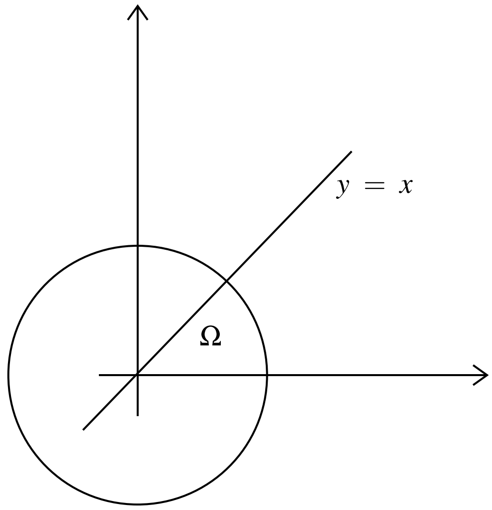
$$\int_a^b f(x)dx = \int_c^d f(g(t))g'(t)dt.$$

Note the extra $g'(t)$ term. Where does it come from? Well, heuristically we have

$$\frac{dx}{dt} = g'(t)dt \Rightarrow dx = g'(t)dt.$$

There is a way to make this rigorous using the $\varepsilon - \delta$ language, but let us accept this for now.

Now suppose that we consider polar coordinates. Say we consider the following sector $\Omega$ which we considered before and say we want to integrate the function $f(x,y) = x$.




Before, we set up the integral using $xy$ coordinates as

$$\int_0^{\sqrt{2}/2} \int_y^{\sqrt{1-y^2}} xdydx = \frac{\sqrt{2}}{6}.$$

In terms of polar coordinates, the region is in fact rectangular: $r$ goes from 0 to 1 and for each $r$, the angle goes from 0 to $\pi/4$, which is independent of $r$. We have the formula

$$dydx = dxdy = rdrd\theta = rd\theta dr.$$

Accepting this, the same integral is set up as

$$\int_0^1 \int_0^{\pi/4} (r\cos\theta)rd\theta dr = \int_0^1 r^2 \Big( \sin\theta \Big|_0^{\pi/4} \Big) dr = \frac{\sqrt{2}}{2} \int_0^1 r^2 dr = \frac{\sqrt{2}}{6}.$$