LECTURE 4

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Now we consider taking integrals over a more general region Ω than just rectangles. However, we can simply reduce to the latter case. Indeed, let $\Omega \subseteq \mathbb{R}^2$ be a certain region and $f : \Omega \to \mathbb{R}$ be a function. Suppose that Ω is bounded and choose a rectangle R containing Ω. Define a function $\widetilde{f} : R \to \mathbb{R}$ by

$$\widetilde{f}(x) = \begin{cases} f(x) \text{ when } x \in R; \\ 0 \text{ otherwise.} \end{cases}$$

Then we say that f is integrable over Ω if $\widetilde{f}$ is integrable over R, and then we set

$$\int_\Omega f := \int_R \widetilde{f}.$$

It is not hard to see that the definition is independent of the choice of R. However, we remark that unlike what we did when defining the integral of f over $[a,b)$, here it is actually important to set $f(x) = 0$ when $x \notin \Omega$, because $R \smallsetminus \Omega$ usually has a positive area.

Theorem 1. (Fubini) *Suppose that Ω is the region*

$$\Omega = \{(x,y) \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\},$$

for some continuous functions $g_1, g_2 : [a,b] \to \mathbb{R}$ with $g_2 \geq g_1$. Then every continuous function $f : \Omega \to \mathbb{R}$ is integrable and

$$\int_\Omega f = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx.$$

Proof. This can be proved by slightly modifying the proof of the case when Ω is a rectangle. □

It is worthwhile reviewing the definition of open and closed subsets.

Definition 2. A subset $\Omega \subseteq \mathbb{R}^n$ is said to be open if for every $x \in \Omega$, there exists some $\varepsilon > 0$ such that for every x' with $\|x - x'\| < \varepsilon$, $x' \in \Omega$. In other words, a small enough open ball containing x is contained in Ω.

Definition 3. A subseteq $\Omega \subseteq \mathbb{R}^n$ is said to be closed if its complement is open.

Here is a handy fact about closed subsets.

Theorem 4. *A subset $\Omega \subseteq \mathbb{R}^n$ is closed if whenever there is a sequence $x_n \in \Omega$ which converges in $\mathbb{R}^n$, the limit is contained in Ω. In other words, Ω contains all its limit points.*

In this class, we will only consider integrals over open or closed subsets (or a mixture like $[a,b)$), we will not consider bezarre examples like $\Omega = \mathbb{Q} \subseteq \mathbb{R}$. You will learn about these things when studying Lebesgue integrals.

Example 5. Suppose that Ω is the triangle with vertices $\{(0,0), (\pi/6, 0), (\pi/6, \pi/6)\}$ and $f : \Omega \smallsetminus \{x = 0\} \to \mathbb{R}$ is defined by $\sin(x)/x$. Let us compute $\int_\Omega f$.

First, we should note that f extends to a continuous function over Ω. Indeed, note that

$$\lim_{x \to 0} \frac{\sin(x)}{x} = \cos'(0) = 1,$$

so that we can simply set $f(x,y) = 1$ if $x = 0$ to obtain a continuous function. The continuity implies that we are allowed to apply Fubini's theorem.

$$\int_0^{\pi/6} \int_0^x \frac{\sin(x)}{x} dydx = \int_0^{\pi/6} \frac{\sin(x)}{x} y\Big|_0^x dx = \int_0^{\pi/6} \sin(x)dx = -\sin(\frac{\pi}{6}) + 1 = \frac{1}{2}.$$

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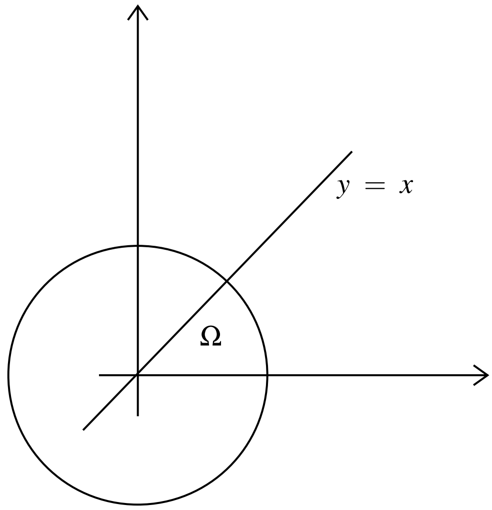
On the other hand, if we try to evaluate the double integral in the other way, then we get stuck, because the indefinite integral of $\sin(x)/x$ does not have an elementary expression.

Example 6. Let Ω be the sector of the unit circle bounded by the arc connecting $(1,0)$ and $(\sqrt{2}/2, \sqrt{2}/2)$. Let us compute $\int_\Omega f$ for $f(x,y) = x$.



If we first integrate in the x-direction and then the y-direction, then we set up the integral as

$$\int_0^{\sqrt{2}/2} \int_y^{\sqrt{1-y^2}} x dx dy = \int_0^{\sqrt{2}/2} \left(\frac{1}{2} x^2 \Big|_y^{\sqrt{1-y^2}} \right) dy = \frac{1}{2} \int_0^{\sqrt{2}/2} 1 - 2y^2 dy = \frac{\sqrt{2}}{6}.$$

If we integrate in the other way, then we do

$$\begin{aligned} \int_0^{\sqrt{2}/2} \int_0^x x dy dx + \int_{\sqrt{2}/2}^1 \int_0^{\sqrt{1-x^2}} x dy dx &= \int_0^{\sqrt{2}/2} x^2 dx + \int_{\sqrt{2}/2}^1 x\sqrt{1-x^2} dx \\ &= \frac{\sqrt{2}}{12} + \left(\frac{-1}{3} (1-x^2)^{3/2} \Big|_{\sqrt{2}/2}^1 \right) = \frac{\sqrt{2}}{6}. \end{aligned}$$