LECTURE 3

ZIQUAN YANG

In calculus you know that if a function f is defined on an interval which is not closed and bounded, say $[a,b)$, sometimes it still makes sense to integrate f. There are two ways you can take to make sense of this. One way is to define

$$\int_a^b f := \lim_{b' \to b^-} \int_a^{b'} f$$

provided that the right hand side exists. Another way is to define a function $\widetilde{f} : [a,b] \to 0$ which agrees with f over $[a,b)$ and $\widetilde{f}(b) = 0$. Then we define

$$\int_a^b f = \int_a^b \widetilde{f}$$

provided that the right hand side exists. The two ways are equivalent. The second way is conceptually easier whereas the first is more useful for computations. For example, we have

$$\int_0^1 \frac{1}{\sqrt{x}} dx = \lim_{a \to 0^+} \int_a^1 \frac{1}{\sqrt{x}} dx = \lim_{a \to 0^+} (1 - \sqrt{a}) = 1$$

although the integrand is not defined at $x = 0$.

A key consequence of Fubini's theorem is that when you integrate a continuous function f over $[a,b] \times [c,d]$, you can exchange of order of integration. This is not true when you have non-closed intervals.

Example 1. Suppose we want to integrate the function

$$f(x,y) = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

over $(0,1] \times (0,1]$. Note that

$$\frac{x^2 - y^2}{(x^2 + y^2)^2} = \frac{\partial}{\partial x}\left(\frac{-x}{x^2 + y^2}\right) = \frac{\partial}{\partial y}\left(\frac{y}{x^2 + y^2}\right).$$

Now we compute

$$\int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} dx dy = \int_0^1 \frac{-x}{x^2 + y^2}\Big|_0^1 dy = \int_0^1 \frac{-1}{1 + y^2} dy = -\tan^{-1}(y)\Big|_0^1 = -\frac{\pi}{4}.$$

However,

$$\int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} dx dy = \int_0^1 \frac{y}{x^2 + y^2}\Big|_0^1 dx = \int_0^1 \frac{1}{1 + x^2} dx = \tan^{-1}(x)\Big|_0^1 = \frac{\pi}{4}.$$

In fact, the function f is not integrable over the region $(0,1] \times (0,1]$ even though both double integrals make sense. In the proof of Fubini's theorem, we used that a continuous function over a closed bounded interval is automatically uniformly continuous. This is not true in general for non-closed intervals.

On the other hand, when Fubini's theorem is applicable, the flexibility of exchanging the order of integration can be a great advantage.

Date: January 15, 2025.

Example 2. Integrate the function $f(x,y) = x\sin(xy)$ over the region $[0,1]\times[\pi/2,\pi]$.

$$\begin{aligned}\int_0^1 x\sin(xy)dx &= \int_0^1 x(\frac{-\cos(xy)}{y})'dx \\ &= x(\frac{-\cos(xy)}{y})\Big|_0^1 - \int_0^1 (\frac{-\cos(xy)}{y})dx \\ &= (\frac{-\cos(y)}{y}) + \frac{\sin(y)}{y^2}\end{aligned}$$

Then we do

$$\int_{\pi/2}^{\pi} (\frac{-\cos(y)}{y}) + \frac{\sin(y)}{y^2} dy,$$

which is very hard. However, if we do

$$\int_0^1 \int_{\pi/2}^{\pi} x\sin(xy)dydx = \int_0^1 -\cos(xy)\Big|_{\pi/2}^{\pi} dx = \frac{2}{\pi},$$

then it becomes much easier.