LECTURE 2

ZIQUAN YANG

Let us first recall what it means for f to be continuous. Suppose that $\Omega \subseteq \mathbb{R}^n$ is any region and $f : \Omega \to \mathbb{R}$ is a function. We say that f is continous on S if for every $x_0 \in S$,

$$\lim_{x \to x_0} f(x) = f(x_0).$$

In words, the meaning is "$f(x)$ is arbitrarily close to $f(x_0)$ as long as x gets close enough to x_0". In $\varepsilon - \delta$ language, the meaning is that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ whenever $\|x - x_0\| < \delta$. Note that here δ depends both on ε and on x_0.

Example 1. Let us show that $f : \mathbb{R} \to \mathbb{R}$ defined by $x \mapsto x^2$ is continous. Take any $x_0 \in \mathbb{R}$ and $\varepsilon > 0$. We want to make sure that δ is such that

$$|(x_0 + \delta)^2 - x_0^2| = |2x_0\delta + \delta^2| < \varepsilon.$$

To make sure that this happens, we may take

$$\delta = \min\{\frac{\varepsilon}{4x_0}, \sqrt{\frac{\varepsilon}{2}}\}.$$

I am being budget on choosing δ. But you can still see that if you fix ε, there is no single δ that works for all x_0.

If for every $\varepsilon > 0$, δ can be chosen independently from x_0, then we say that f is *uniformly continous*.

> Fact: If the domain of definition Ω is closed and bounded (e.g., a product of closed intervals), then any continous f is automatically uniformly continous.

Let us make a simple obvervation on estimates.

Lemma 2. *Suppose that I is an interval of length ℓ and $f : I \to \mathbb{R}$ is an integrable function such that $|f(x) - c| < \varepsilon$ for every $x \in I$. Then*

$$|(\int_I f) - c\ell| < \varepsilon\ell.$$

More generally, we can replace I by a product of closed intervals $\Omega \subseteq \mathbb{R}^n$ and ℓ by the volume.

Proof. Note that

$$|(\int_I f) - c\ell| = |\int_I (f - c)| \leq \int_I |f - c| \leq \int_I \varepsilon = \varepsilon\ell.$$

□

> **Theorem 3.** (Fubini) *Set $\Omega := [a,b] \times [c,d] \subseteq \mathbb{R}^2$ Let $f : \Omega \to \mathbb{R}$ be a continuous function. Then*
>
> $$\int_\Omega f = \int_c^d \Big(\int_a^b f(x,y)dx\Big)dy = \int_a^b \Big(\int_c^d f(x,y)dy\Big)dx.$$

Proof. It suffices to prove the first equality, as the second is obtained by switching the role of x and y.

Choose an arbitrary $\varepsilon > 0$. By the uniform continuity of f, there exists a $\delta > 0$ such that whenever $\|P - P'\| < \delta$, $|f(P) - f(P')| < \varepsilon$ for two points $P, P' \in \Omega$. Let $P_x = \{a = x_0, x_1, \cdots, x_n = b\}$ and $P_y = \{c = y_0, y_1, \cdots, y_m = d\}$ be partitions of $[a,b]$ and $[c,d]$ respectively, and let P be the partition of Ω by the corresponding grid. Let $R_{i,j}$ be the rectangle whose bottom left corner is (x_i, y_j). Choose P_x and

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P_y to be small enough such that the diagonal of each $R_{i,j}$ is $< \delta$. Then for every $(x,y) \in R_{i,j}$, we have $|f(x,y) - f(x_i,y_j)| < \varepsilon$ by construction. By the lemma, we know that

$$|f(x_i,y_j)\text{Area}(R_{i,j}) - \int_{R_{i,j}} f| < \varepsilon\text{Area}(R_{i,j}).$$

Therefore,

$$|L(P,f) - \int_{\Omega} f| \leq \sum_{i,j} |f(x_i,y_j)\text{Area}(R_{i,j}) - \int_{R_{i,j}} f| < \sum_{i,j} \varepsilon\text{Area}(R_{i,j}) = \varepsilon\text{Area}(\Omega).$$

On the other hand, let $F(y) = \int_{x_i}^{x_{i+1}} f(x,y)dx$. By the lemma, we have that for $(x,y) \in R_{i,j}$,

$$|F(y) - f(x_i,y_j)(x_{i+1} - x_i)| < \varepsilon(x_{i+1} - x_i).$$

Hence by the lemma again,

$$|\int_{y_j}^{y_{j+1}} F(y)dy - f(x_i,y_j)(x_{i+1} - x_i)(y_{j+1} - y_j)| < \varepsilon(x_{i+1} - x_i)(y_{j+1} - y_j).$$

Note that $(x_{i+1} - x_i)(y_{j+1} - y_j) = \text{Area}(R_{i,j})$. This implies that within each $R_{i,j}$, we have

$$|\int_{y_j}^{y_{j+1}} \int_{x_i}^{x_{i+1}} f(x,y)dxdy - f(x_i,y_j)\text{Area}(R_{i,j})| < \varepsilon\text{Area}(R_{i,j}).$$

Add these possible errors for all i, j, we have

$$|\int_c^d \big(\int_a^b f(x,y)dx\big)dy - L(P,f)| < \varepsilon\text{Area}(\Omega).$$

By triangle inequality, we conclude that

$$|\int_c^d \big(\int_a^b f(x,y)dx\big)dy - \int_{\Omega} f| < 2\varepsilon\text{Area}(\Omega).$$

Now, note that the left hand side is independent of ε whereas the right hand side is. The only way for this to be true is that the left hand side is 0. □

Exercise 4. Compute $\int\int_R xy^2 dxdy$ for $R = [0,2] \times [0,1]$. Answer: 2/3.

Remark 5. Note that Fubini's theorem in particular says that if f is continous, then you can switch the order of taking the double integral. Namely, you can integrate with respect to x and then y or do it in the reverse order. This is true for a class of functions which are more general than continous functions. However, f needs to be at least integrable. Otherwise we would have a counterexample.

Before talking about this example, let us take a look at some examples of non-integrable functions.

Example 6. Suppose that $f : [0,1] \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 0 \text{ if } x \in \mathbb{Q}; \\ 1 \textit{ if } x \notin \mathbb{Q}. \end{cases}$$

Then f is not integrable. Indeed, suppose that f is integrable and

$$\int_0^1 f = \lim_{\|P\| \to 0} L(P,f) = L$$

for some L. More concretely, for every $\varepsilon > 0$, there exists some $\delta > 0$ such that for every $\|P\| < \delta$, $|L(P,f) - L| < \varepsilon$. Let us show that it cannot be the case. Fix some $\varepsilon < 1/4$ and take the corresponding δ. We may assume that $\delta < 1/3$. We can always choose a partition P such that $\|P\| < \delta$ and $P \subseteq \mathbb{Q}$, and another partition $P' = \{0 = x'_0, x'_1, \cdots, x'_n = 1\}$ such that $\|P'\| < \delta$ but $x'_i \notin \mathbb{Q}$ for every $0 < i < n$. Then it is easy to see that $L(P,f) = 0$ but $L(P',f) > 2/3$. Now we have

$$|L(P,f) - L| < \varepsilon < 1/4, |L(P',f) - L| < \varepsilon < 1/4, L(P,f) = 0, L(P',f) > 2/3,$$

which contradicts the triangle inequality.[1]

[1] In class, I forgot that P' needs to start with 0 as well and $0 \in \mathbb{Q}$, so here I made a minor correction of the argument.

Example 7. Suppose that $f:[0,1]\to\mathbb{R}$ is defined by

$$f(x)=\begin{cases}0 \text{ if } x=0;\\ 1/x \textit{ if } x>0.\end{cases}$$

Then f is not integrable. One way you can do this is to consider the partition $P_n := \{0,1/n,2/n,\cdots 1\}$. Then

$$L(P_n,f)=\sum_{i=1}^{n-1}(\frac{i}{n})^{-1}\frac{1}{n}=\sum_{i=1}^{n-1}\frac{1}{i}.$$

Since we know that the series

$$1+\frac{1}{2}+\frac{1}{3}+\cdots$$

diverges, there is no limit of $L(P_n,f)$. If you do not want to make use of this hard fact that the above series diverges (which is not obvious), you may argue as follows. In fact, it is easy to see that for every $\delta>0$, there exists some P such that $\|P\|<\delta$ but $L(P,f)$ gets arbitrary big; more precisely, for every fixed δ and an arbitrary M, there exists some P such that $\|P\|<\delta$ but $L(P,f)>M$. Indeed, fix some $\delta'<\delta$, suppose that we take a partition $P=\{0,x_1,x_2,\cdots,1\}$ with $\|P\|<\delta$ such that $x_1=\delta'/M$, $x_2=x_1+\delta'$. Then $L(P,f)>f(x_1)(x_2-x_1)=M$.