LECTURE 1

ZIQUAN YANG

Recall how integrals are defined for a single variable function. Let $f : [a,b] \to \mathbb{R}$ be a function. By a partition P of $[a,b]$ we mean a sequence of numbers $a = x_0, \cdots, x_n = b$ with endpoints being a and b. By the norm $\|P\|$ of P, we mean the maximum of the all the intervals in P. For each P, we set

$$L(P, f) = \sum_{i=0}^{n-1} f(x_i)(x_{i+1} - x_i).$$

Definition 1. We say that f is *integrable* if the limit

$$\lim_{\|P\| \to 0} L(P, f) \tag{1}$$

exists, in which case we define the limit to be the integral $\int_a^b f$.

In the definition, I evaluated f at the left endpoint x_i of each interval $[x_i, x_{i+1}]$. This choice is not important. If you like, you can also use the right endpoint x_{i+1}, or any point in between.[1]

Theorem 2. *If f is continuous on $[a,b]$, then it is integrable.*

Once you know that the limit (1) exists, it does not matter how you choose to approximate it. For example, you can consider an arbitrary sequence of partitions $P_1, P_2, \cdots$ such that $\|P_n\| \to 0$ as $n \to \infty$. Then the limit is the limit of $L(P_n, f)$.

Example 3. Let us compute $\int_0^1 x^2 dx$ using Riemann sums. It is often easiest when we use uniform partition, namely those whose intervals are of equal length. Therefore, let P_n be the partition $\{0, 1/n, 2/n, \cdots, 1\}$. Then

$$L(P_n, f) = \sum_{i=0}^{n-1} (\frac{i}{n})^2 \cdot \frac{1}{n} = \frac{1}{n^3} \sum_{i=0}^{n-1} i^2 = \frac{(n-1)n(2n-1)}{6n^3}.$$

By comparing the leading terms of the numerator and denominator, we easily see that

$$\lim_{n \to \infty} \frac{(n-1)n(2n-1)}{6n^3} = \frac{1}{3},$$

as expected.

Now we move to dimension 2. Suppose now that $S = [a,b] \times [c,d]$ is a rectangle in $\mathbb{R}^2$ and $f : S \to \mathbb{R}$ is a function. Let P be a partition of S into smaller rectangles. It does not need to be as regular as a grid—any division into smaller rectangles works. For each smaller rectangle $R \in P$, let us write c_R for the point at the bottom left corner. Again, the choice of this point does not matter. Then we define

$$L(P, f) = \sum_{R \in P} f(c_R) \cdot \text{Area}(R).$$

Define $\|f\|$ to be the maximum of $\{\text{Area}(R) : R \in P\}$. Then Definition 1 makes sense in dimension 2 verbatim. If f is integrable, we write its integral as

$$\int_S f \text{ or } \int_c^d \int_a^b f(x,y)dxdy.$$

In fact, one needs to be careful with the second notation. However, explaining the caveats is beyond the scope of the course. For continuous functions, which we mostly care about in the class, this is harmless.

Date: January 8, 2025.

[1] If you are annoyed by this "choice of points", you are rightfully so. The mathematically rigorous way of saying this should involve sup and inf.