HOMEWORK 11

NAME:____________________

Problem 1: Let $\mathbf{F} = (y\cos 2x)\mathbf{i} + (y^2 \sin 2x)\mathbf{j} + (x^2y + z)\mathbf{k}$. Is there a vector field $\mathbf{A}$ such that $\mathbf{F} = \nabla \times \mathbf{A}$? Explain your answer.

Date: April 16, 2025.

Problem 2: Compute the net outward flux of the vector field

$$\mathbf{F} = \frac{(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})}{(x^2 + y^2 + z^2)^{3/2}}$$

across the ellipsoid $9x^2 + 4y^2 + 6z^2 = 36$.

Problem 3: Find the flux of the vector field:

$$\mathbf{F} = (5x^3 + 12xy^2)\mathbf{i} + (y^3 + e^y \sin z)\mathbf{j} + (5z^3 + e^y \cos z)\mathbf{k},$$

through the boundary of the region D as the solid between the spheres:

$$x^2 + y^2 + z^2 = 1 \quad \text{and} \quad x^2 + y^2 + z^2 = 2.$$

Problem 4: A function $f(x,y,z)$ is said to be *harmonic* in a region D in space if it satisfies the *Laplace equation*

$$\nabla^2 f = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

throughout D.

(a) Suppose that f is harmonic throughout a bounded region D enclosed by a smooth surface S and that $\mathbf{n}$ is the chosen unit normal vector on S. Show that the integral over S of $\nabla f \cdot \mathbf{n}$, the derivative of f in the direction of $\mathbf{n}$, is zero.

(b) Show that if f is harmonic on D, then

$$\iint_S f \nabla f \cdot \mathbf{n}\, d\sigma = \iiint_D \|\nabla f\|^2 \, dV.$$

Problem 5: Suppose that f and g are scalar functions with continuous first- and second-order partial derivatives throughout a region D that is bounded by a closed piecewise smooth surface S. Show that

$$\iint_S f\nabla g \cdot \mathbf{n}\, d\sigma = \iiint_D \left(f\nabla^2 g + \nabla f \cdot \nabla g\right) dV.$$

The above equation is *Green's first formula*. (*Hint:* Apply the Divergence Theorem to the field $\mathbf{F} = f\nabla g$.)