Suggested Solution to HOMEWORK 6

Problem 1

Solution.

$$\begin{aligned} \vec{r}(t) &= (t^2, t^2, t^2) \\ \vec{r}'(t) &= (2t, 2t, 2t) \\ \|\vec{r}'(t)\| &= \sqrt{(2t)^2 + (2t)^2 + (2t)^2} = 2\sqrt{3}t \\ f(\vec{r}(t)) &= t^2 + t - t^4 \end{aligned}$$

$$\begin{aligned} \int_C f ds = &\int_0^1 \left(t^2 + t - t^4\right) 2\sqrt{3} t dt \\ =& 2\sqrt{3} \int_0^1 (t^3 + t^2 - t^5) dt \\ =& 2\sqrt{3} \left[\frac{t^4}{4} + \frac{t^3}{3} - \frac{t^6}{6}\right]_0^1 \\ =& 2\sqrt{3} \left(\frac{1}{4} + \frac{1}{3} - \frac{1}{6}\right) \\ =& \frac{5\sqrt{3}}{6} \end{aligned}$$

□

Problem 2

Solution.

$$\begin{aligned} r(\theta) &= 2 + 2\cos\theta, 0 \le \theta \le 2\pi \\ \vec{r}(\theta) &= (x, y) = (r(\theta)\cos\theta, r(\theta)\sin\theta) \\ \vec{r}'(\theta) &= (-r(\theta)\sin\theta + r'(\theta)\cos\theta, r(\theta)\cos\theta + r'(\theta)\sin\theta) \\ \|\vec{r}'(t)\| &= \sqrt{(r(\theta))^2 + (r'(\theta))^2} = \sqrt{(2 + 2\cos\theta)^2 + (-2\sin\theta)^2} \end{aligned}$$

$$\begin{aligned} \text{Arc length} = \int_C 1 = &\int_0^{2\pi} \sqrt{(2 + 2\cos\theta)^2 + (-2\sin\theta)^2} d\theta \\ =& \int_0^{2\pi} \sqrt{4 + 8\cos\theta + 4\cos^2\theta + 4\sin^2\theta} d\theta \\ =& \int_0^{2\pi} \sqrt{8 + 8\cos\theta} d\theta \\ =& 2\int_0^{\pi} \sqrt{8\left(2\cos^2\frac{\theta}{2}\right)} d\theta \\ =& 8\int_0^{\pi} \cos\frac{\theta}{2} d\theta \quad (\cos\frac{\theta}{2} \ge 0 \text{ for } 0 \le \theta \le \pi) \\ =& 8\left[2\sin\frac{\theta}{2}\right]_0^{\pi} \\ =& 16 \end{aligned}$$

□

Problem 3

Solution.
$F = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is defined for all $(x, y, z) \in \mathbb{R}^3$.
F is on a simply connected open domain of $\mathbb{R}^3$. Also, It is continuously differentiable.

$$\frac{\partial x}{\partial y} = 0 = \frac{\partial y}{\partial x}$$

$$\frac{\partial x}{\partial z} = 0 = \frac{\partial z}{\partial x}$$

$$\frac{\partial y}{\partial z} = 0 = \frac{\partial z}{\partial y}$$

Hence, F is conservative.
Since C is a closed curve, the circulation of the field F around C in either orientation is zero. □

Problem 4

Solution.
$C_1 : \vec{r_1}(t) = (1, -1) + t(0, 1) = (1, -1 + t), 0 \le t \le 2$
$C_2 : \vec{r_2}(t) = (1, 1) + t(-1, 0) = (1 - t, 1), 0 \le t \le 2$
$C_3 : \vec{r_3}(t) = (-1, 1) + t(0, -1) = (-1, 1 - t), 0 \le t \le 2$
$C_4 : \vec{r_4}(t) = (-1, -1) + t(1, 0) = (-1 + t, -1), 0 \le t \le 2$
$\mathbf{F}(\vec{r_1}(t)) = (1, 1 - t)$
$\mathbf{F}(\vec{r_2}(t)) = (1 - t, -1)$
$\mathbf{F}(\vec{r_3}(t)) = (-1, -1 + t)$
$\mathbf{F}(\vec{r_4}(t)) = (-1 + t, 1)$
$\vec{r_1}'(t) = (0, 1)$
$\vec{r_2}'(t) = (-1, 0)$
$\vec{r_3}'(t) = (0, -1)$
$\vec{r_4}'(t) = (1, 0)$

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\vec{r} &= \int_{C_1} \mathbf{F} \cdot d\vec{r_1} + \int_{C_2} \mathbf{F} \cdot d\vec{r_2} + \int_{C_3} \mathbf{F} \cdot d\vec{r_3} + \int_{C_4} \mathbf{F} \cdot d\vec{r_4} \\
&= \int_0^2 (1, 1 - t) \cdot (0, 1) dt + \int_0^2 (1 - t, -1) \cdot (-1, 0) dt + \int_0^2 (-1, -1 + t) \cdot (0, -1) dt + \int_0^2 (-1 + t, 1) \cdot (1, 0) dt \\
&= \int_0^2 1 - t dt + \int_0^2 -1 + t dt + \int_0^2 1 - t dt + \int_0^2 -1 + t dt \\
&= \int_0^2 0 dt \\
&= 0
\end{aligned}$$

□

Problem 5

Solution.
$C_1 : \vec{r_1}(t) = (0,0) + t((1,3) - (0,0)) = (t, 3t), 0 \le t \le 1$
$\sqrt{x+y} = \sqrt{t+3t} = 2\sqrt{t}$
$dx = dt$

$C_2 : \vec{r_2}(t) = (1,3) + t((0,3) - (1,3)) = (1-t, 3), 0 \le t \le 1$
$\sqrt{x+y} = \sqrt{1-t+3} = \sqrt{4-t}$
$dx = -dt$

$C_3 : \vec{r_3}(t) = (0,3) + t((0,0) - (0,3)) = (0, 3-3t), 0 \le t \le 1$
$\sqrt{x+y} = \sqrt{0+3-3t} = \sqrt{3-3t}$
$dx = 0$

$$\begin{aligned}
\int_C \sqrt{x+y}dx &= \int_{C_1} \sqrt{x+y}dx + \int_{C_2} \sqrt{x+y}dx + \int_{C_3} \sqrt{x+y}dx \\
&= \int_0^1 2\sqrt{t}dt + \int_0^1 \sqrt{4-t}(-1)dt \\
&= \int_0^1 2\sqrt{t}dt + \int_0^1 \sqrt{4-t}d(4-t) \\
&= 2\left[\frac{2}{3}t^{\frac{3}{2}}\right]_0^1 + \left[\frac{2}{3}(4-t)^{\frac{3}{2}}\right]_0^1 \\
&= \frac{4}{3} + \frac{2}{3}3\sqrt{3} - \frac{2}{3}8 \\
&= 2\sqrt{3} - 4
\end{aligned}$$

□