Suggested Solution to HOMEWORK 5

Problem 1

Solution.

$$x = u/v, y = uv$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \det\begin{bmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{bmatrix} = \frac{2u}{v}$$

$$xy = 1 \Rightarrow \frac{u}{v}uv = 1 \Rightarrow u^2 = 1 \Rightarrow u = 1$$

$$xy = 9 \Rightarrow \frac{u}{v}uv = 9 \Rightarrow u^2 = 9 \Rightarrow u = 3$$

$$y = x \Rightarrow uv = \frac{u}{v} \Rightarrow v^2 = 1 \Rightarrow v = 1$$

$$y = 4x \Rightarrow uv = 4\frac{u}{v} \Rightarrow v^2 = 4 \Rightarrow v = 2$$

$$\begin{aligned}
&\iint_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy}\right) dxdy \\
=& \int_1^2 \int_1^3 (v+u)\frac{2u}{v} dudv \\
=&2 \int_1^2 \int_1^3 u + \frac{u^2}{v} dudv \\
=&2 \int_1^2 \left[\frac{u^2}{2} + \frac{u^3}{3v}\right]_1^3 dv \\
=&2 \int_1^2 4 + \frac{26}{3v} dv \\
=&2 \left[4v + \frac{26}{3}\ln v\right]_1^2 \\
=&2 \left(8 + \frac{26}{3}\ln 2 - 4\right) \\
=&8 + \frac{52}{3}\ln 2
\end{aligned}$$

□

Problem 2

Solution.

$$u = x - y/2, v = y$$
$$\Rightarrow x = u + \frac{v}{2}, y = v$$
$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \det\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix} = 1$$
$$y = 0 \Rightarrow v = 0$$
$$y = 1/2 \Rightarrow v = \frac{1}{2}$$
$$x = y/2 \Rightarrow u + \frac{v}{2} = \frac{v}{2} \Rightarrow u = 0$$
$$x = (y+4)/2 \Rightarrow u + \frac{v}{2} = \frac{v+4}{2} \Rightarrow u = 2$$

$$\begin{aligned}
&\int_0^{1/2}\int_{y/2}^{(y+4)/2} y^3(2x-y)e^{(2x-y)^2}dxdy \\
=&\int_0^{\frac{1}{2}}\int_0^2 v^3(2u)e^{(2u)^2}dudv \\
=&\int_0^{\frac{1}{2}}\int_0^2 v^3\frac{1}{4}e^{(2u)^2}d((2u)^2)dv \\
=&\int_0^{\frac{1}{2}} \frac{v^3}{4}\left[e^{(2u)^2}\right]_0^2 dv \\
=&\int_0^{\frac{1}{2}} \frac{v^3}{4}(e^{16}-1)dv \\
=&\frac{e^{16}-1}{4}\left[\frac{v^4}{4}\right]_0^{\frac{1}{2}} \\
=&\frac{e^{16}-1}{4}\frac{1}{64} \\
=&\frac{e^{16}-1}{256}
\end{aligned}$$

□

Problem 3

Solution.

$$u = x, v = xy, w = 3z$$
$$\Rightarrow x = u, y = \frac{v}{u}, z = \frac{w}{3}$$
$$\det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix} = \det\begin{bmatrix} 1 & 0 & 0 \\ -\frac{v}{u^2} & \frac{1}{u} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} = \frac{1}{3u}$$
$$x = 1 \Rightarrow u = 1$$
$$x = 2 \Rightarrow u = 2$$
$$xy = 0 \Rightarrow v = 0$$
$$xy = 1 \Rightarrow v = 1$$
$$z = 0 \Rightarrow w = 0$$
$$z = 2 \Rightarrow w = 6$$

$$\begin{aligned}
\text{Mass} &= \int_0^6 \int_0^1 \int_1^2 \left(uv + \frac{2}{3}vw\right)\frac{1}{3u}dudvdw \\
&= \int_0^6 \int_0^1 \int_1^2 \frac{v}{3} + \frac{2vw}{9u}dudvdw \\
&= \int_0^6 \int_0^1 \frac{v}{3} + \frac{2vw}{9}\ln 2dvdw \\
&= \int_0^6 \left(\frac{1}{3} + \frac{2w\ln 2}{9}\right)\left[\frac{v^2}{2}\right]_0^1 dw \\
&= \int_0^6 \frac{1}{6} + \frac{w\ln 2}{9}dw \\
&=1 + \frac{\ln 2}{9}\left[\frac{w^2}{2}\right]_0^6 \\
&=1 + 2\ln 2
\end{aligned}$$

□

Problem 4

Solution.
Note that we have $rdzdrd\theta = dxdydz = \rho^2 \sin\phi d\rho d\phi d\theta$.

$$
\begin{aligned}
&\int_0^{2\pi} \int_0^{\sqrt{2}} \int_r^{\sqrt{4-r^2}} 3rdzdrd\theta \\
=&\int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{2} 3\rho^2 \sin\phi d\rho d\phi d\theta \\
=&\int_0^{2\pi} \int_0^{\frac{\pi}{4}} \left[\rho^3 \sin\phi\right]_0^2 d\phi d\theta \\
=&\int_0^{2\pi} \int_0^{\frac{\pi}{4}} 8\sin\phi d\phi d\theta \\
=&8\int_0^{2\pi} \left[-\cos\phi\right]_0^{\frac{\pi}{4}} d\theta \\
=&8\int_0^{2\pi} -\frac{\sqrt{2}}{2} + 1 d\theta \\
=&8\left(-\frac{\sqrt{2}}{2} + 1\right)(2\pi) \\
=&(16 - 8\sqrt{2})\pi
\end{aligned}
$$

□

Problem 5

Solution.

$$x = u^2 - v^2, y = 2uv$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \det\begin{bmatrix} 2u & -2v \\ 2v & 2u \end{bmatrix} = 4u^2 + 4v^2$$

$$y = 0 \Rightarrow 2uv = 0 \Rightarrow u = 0 \text{ or } v = 0$$

$$y = 2\sqrt{1-x} \Rightarrow 2uv = 2\sqrt{1-(u^2-v^2)} \Rightarrow u^2v^2 = 1 - u^2 + v^2 \Rightarrow (u^2-1)(v^2+1) = 0 \Rightarrow u = 1 \text{ or } -1$$

$$x = 0 \Rightarrow u^2 - v^2 = 0 \Rightarrow u^2 = v^2 \Rightarrow u = v \text{ (since } y = 2uv \geq 0)$$

$$x = 1 \Rightarrow u^2 - v^2 = 1 \Rightarrow u^2 = 1 + v^2 \geq 1$$

Take $u \geq 0$, the region is the triangle bounded by $v = 0, u = v, u = 1$.

$$\begin{aligned}
&\int_0^1 \int_0^{2\sqrt{1-x}} \sqrt{x^2+y^2}\,dydx \\
=&\int_0^1 \int_0^u \sqrt{(u^2-v^2)^2+(2uv)^2}(4u^2+4v^2)dvdu \\
=&\int_0^1 \int_0^u (u^2+v^2)(4u^2+4v^2)dvdu \\
=&4\int_0^1 \int_0^u u^4 + 2u^2v^2 + v^4 dvdu \\
=&4\int_0^1 u^5 + 2u^2\left[\frac{v^3}{3}\right]_0^u + \left[\frac{v^5}{5}\right]_0^u du \\
=&4\int_0^1 u^5 + \frac{2}{3}u^5 + \frac{1}{5}u^5 du \\
=&4\int_0^1 \frac{28}{15}u^5 du \\
=&4\left(\frac{28}{15}\right)\left[\frac{u^6}{6}\right]_0^1 \\
=&4\left(\frac{28}{15}\right)\left(\frac{1}{6}\right) \\
=&\frac{56}{45}
\end{aligned}$$

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