# Suggested Solution to HOMEWORK 4

## Problem 1

*Solution.*

$$
\begin{aligned}
&\int_{-1}^{1}\int_{0}^{2\pi}\int_{0}^{1+\cos(\theta)} 4r\,dr\,d\theta\,dz\\
=&\int_{-1}^{1}\int_{0}^{2\pi}\left[2r^2\right]_0^{1+\cos(\theta)} d\theta\,dz\\
=&\int_{-1}^{1}\int_{0}^{2\pi} 2(1+\cos(\theta))^2 d\theta\,dz\\
=&\int_{-1}^{1}\int_{0}^{2\pi} (2+4\cos(\theta)+2\cos^2(\theta))d\theta\,dz\\
=&\int_{-1}^{1}\int_{0}^{2\pi} (2+4\cos(\theta)+\cos(2\theta)+1)d\theta\,dz\\
=&\int_{-1}^{1}\left[3\theta+4\sin(\theta)+\frac{1}{2}\sin(2\theta)\right]_0^{2\pi} dz\\
=&\int_{-1}^{1} 6\pi\,dz\\
=&12\pi
\end{aligned}
$$

□

## Problem 2

*Solution.*
$x = r\cos\theta, y = r\sin\theta$.

$$
\begin{aligned}
&\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}}\int_{0}^{x} (x^2+y^2)\,dz\,dx\,dy\\
=&\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{0}^{1}\int_{0}^{r\cos\theta} (r^2)r\,dz\,dr\,d\theta\\
=&\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{0}^{1} r^3 r\cos\theta\,dr\,d\theta\\
=&\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{0}^{1} r^4\cos\theta\,dr\,d\theta\\
=&\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\frac{r^5}{5}\cos\theta\right]_0^1 d\theta\\
=&\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{5}\cos\theta\,d\theta\\
=&\left[\frac{1}{5}\sin\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\\
=&\frac{1}{5}-\left(-\frac{1}{5}\right)\\
=&\frac{2}{5}
\end{aligned}
$$

$\square$

## Problem 3

*Solution.*

$$
\begin{aligned}
&\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \int_{\csc(\phi)}^{2\csc(\phi)} \int_0^{2\pi} \rho^2 \sin(\phi) d\theta d\rho d\phi \\
=&2\pi \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \int_{\csc(\phi)}^{2\csc(\phi)} \rho^2 \sin(\phi) d\rho d\phi \\
=&\frac{2\pi}{3} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left[\rho^3 \sin(\phi)\right]_{\csc(\phi)}^{2\csc(\phi)} d\phi \\
=&\frac{2\pi}{3} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 7\csc^3(\phi)\sin(\phi) d\phi \\
=&\frac{14\pi}{3} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \csc^2(\phi) d\phi \\
=&\frac{14\pi}{3} \left[-\cot(\phi)\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} \\
=&\frac{14\pi}{3} \left(-\frac{1}{\sqrt{3}} + \sqrt{3}\right) \\
=&\frac{14\pi}{3} \frac{2}{\sqrt{3}} \\
=&\frac{28\pi}{3\sqrt{3}} = \frac{28\sqrt{3}\pi}{9}
\end{aligned}
$$

$\square$

## Problem 4

*Solution.*

$$
\begin{aligned}
&\int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_1^{1+\cos(\phi)} \rho^2 \sin(\phi) d\rho d\phi d\theta \\
=&\int_0^{2\pi} \int_0^{\frac{\pi}{2}} \left[\frac{\rho^3}{3} \sin(\phi)\right]_1^{1+\cos(\phi)} d\phi d\theta \\
=&\frac{1}{3} \int_0^{2\pi} \int_0^{\frac{\pi}{2}} ((1+\cos(\phi))^3 \sin(\phi) - \sin(\phi)) d\phi d\theta \\
=&\frac{1}{3} \int_0^{2\pi} \left(-\int_0^{\frac{\pi}{2}} ((1+\cos(\phi))^3 d((1+\cos(\phi)) - \int_0^{\frac{\pi}{2}} \sin(\phi)) d\phi\right) d\theta \\
=&-\frac{1}{3} \int_0^{2\pi} \left[\frac{((1+\cos(\phi))^4}{4} - \cos(\phi)\right]_0^{\frac{\pi}{2}} d\theta \\
=&-\frac{1}{3} \int_0^{2\pi} \left(\frac{1}{4} - \frac{2^4}{4} + 1\right) d\theta \\
=&\frac{1}{3} \int_0^{2\pi} \frac{11}{4} d\theta \\
=&\frac{11}{12}(2\pi) \\
=&\frac{11}{6}\pi
\end{aligned}
$$

□

## Problem 5

*Solution.*

$$
\begin{aligned}
&\int_0^{2\pi}\int_1^{\sqrt{2}}\int_{-\sqrt{2-r^2}}^{\sqrt{2-r^2}} r\,dz\,dr\,d\theta \\
&= \int_0^{2\pi}\int_1^{\sqrt{2}} 2r\sqrt{2-r^2}\,dr\,d\theta \\
&= \int_0^{2\pi}\int_1^{\sqrt{2}} -\sqrt{2-r^2}\,d(2-r^2)\,d\theta \\
&= \int_0^{2\pi}\left[-\frac{2}{3}(2-r^2)^{\frac{3}{2}}\right]_1^{\sqrt{2}} d\theta \\
&= \int_0^{2\pi}\frac{2}{3}d\theta \\
&= \frac{2}{3}(2\pi) \\
&= \frac{4\pi}{3}
\end{aligned}
$$

or

$$
\begin{aligned}
&\int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\int_{\frac{1}{\sin\phi}}^{\sqrt{2}} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta \\
&= \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\left[\frac{\rho^3}{3}\sin\phi\right]_{\frac{1}{\sin\phi}}^{\sqrt{2}} d\phi\,d\theta \\
&= \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\left(\frac{2\sqrt{2}}{3}\sin\phi-\frac{1}{3\sin^2\phi}\right)d\phi\,d\theta \\
&= \int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\left(\frac{2\sqrt{2}}{3}\sin\phi-\frac{1}{3}\csc^2\phi\right)d\phi\,d\theta \\
&= \int_0^{2\pi}\left[-\frac{2\sqrt{2}}{3}\cos\phi+\frac{1}{3}\cot\phi\right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} d\theta \\
&= \int_0^{2\pi}\left(-\frac{2\sqrt{2}}{3}\left(-\frac{1}{\sqrt{2}}\right)+\frac{1}{3}(-1)+\frac{2\sqrt{2}}{3}\frac{1}{\sqrt{2}}-\frac{1}{3}(1)\right)d\theta \\
&= \int_0^{2\pi}\frac{2}{3}d\theta \\
&= \frac{2}{3}(2\pi) \\
&= \frac{4\pi}{3}
\end{aligned}
$$

□