Suggested Solution to HOMEWORK 2

Problem 1

Solution.

$$
\begin{aligned}
&\int_0^{\sqrt{3}} \int_{\frac{\sqrt{3}x}{3}}^{\sqrt{4-x^2}} \sqrt{4-x^2}dydx \\
=&\int_0^{\sqrt{3}} \sqrt{4-x^2}(\sqrt{4-x^2}-\frac{\sqrt{3}x}{3})dx \\
=&\int_0^{\sqrt{3}} 4-x^2-\frac{\sqrt{3}x\sqrt{4-x^2}}{3}dx \\
=&\left[4x-\frac{x^3}{3}\right]_0^{\sqrt{3}}+\frac{\sqrt{3}}{3}\int_0^{\sqrt{3}} \frac{\sqrt{4-x^2}}{2}d(4-x^2) \\
=&\left[4x-\frac{x^3}{3}\right]_0^{\sqrt{3}}+\frac{\sqrt{3}}{3}\left[\frac{(4-x^2)^{3/2}}{3}\right]_0^{\sqrt{3}} \\
=&4\sqrt{3}-\frac{3\sqrt{3}}{3}+\frac{\sqrt{3}}{9}-\frac{8\sqrt{3}}{9}=\frac{20\sqrt{3}}{9}
\end{aligned}
$$

or

$$
\begin{aligned}
&\int_{\pi/6}^{\pi/2} \int_0^2 \sqrt{4-(r\cos\theta)^2}rdrd\theta \\
=&\int_{\pi/6}^{\pi/2} \frac{-1}{2\cos^2\theta}\int_0^2 \sqrt{4-(r\cos\theta)^2}d(4-r^2\cos^2\theta)d\theta \\
=&\int_{\pi/6}^{\pi/2} \frac{-1}{3\cos^2\theta}\left[(4-r^2\cos^2\theta)^{3/2}\right]_0^2 d\theta \\
=&\int_{\pi/6}^{\pi/2} \frac{-1}{3\cos^2\theta}(8\sin^3\theta-8)d\theta \\
=&\frac{8}{3}\int_{\pi/6}^{\pi/2} \sec^2\theta-\sin\theta\tan^2\theta d\theta \\
=&\frac{8}{3}\int_{\pi/6}^{\pi/2} \sec^2\theta-\sin\theta\sec^2\theta+\sin\theta d\theta \\
=&\frac{8}{3}\int_{\pi/6}^{\pi/2} \sec^2\theta-\sec\theta\tan\theta+\sin\theta d\theta \\
=&\frac{8}{3}\left[\tan\theta-\sec\theta-\cos\theta\right]_{\pi/6}^{\pi/2} \\
=&\frac{8}{3}\left(\lim_{\theta\to\pi/2}\frac{\sin\theta-1}{\cos\theta}-0-\frac{1}{\sqrt{3}}+\frac{2}{\sqrt{3}}+\frac{\sqrt{3}}{2}\right) \\
=&\frac{8}{3}\left(\lim_{\theta\to\pi/2}\frac{\cos\theta}{-\sin\theta}-0-\frac{1}{\sqrt{3}}+\frac{2}{\sqrt{3}}+\frac{\sqrt{3}}{2}\right) \\
=&\frac{8}{3}\left(0-0-\frac{1}{\sqrt{3}}+\frac{2}{\sqrt{3}}+\frac{\sqrt{3}}{2}\right)=\frac{20\sqrt{3}}{9}
\end{aligned}
$$

□

Problem 2

Solution.

$$\begin{aligned}
&\int_1^2 \int_{-1/x}^{1/x} \int_0^{x+1} 1 dz dy dx \\
=&\int_1^2 \int_{-1/x}^{1/x} x + 1 dy dx \\
=&\int_1^2 (x+1)\frac{2}{x} dx \\
=&\int_1^2 2 + \frac{2}{x} dx \\
=&\left[2x + 2\ln x\right]_1^2 \\
=&4 + 2\ln 2 - 2 - 2\ln 1 \\
=&2 + 2\ln 2
\end{aligned}$$

□

Problem 3

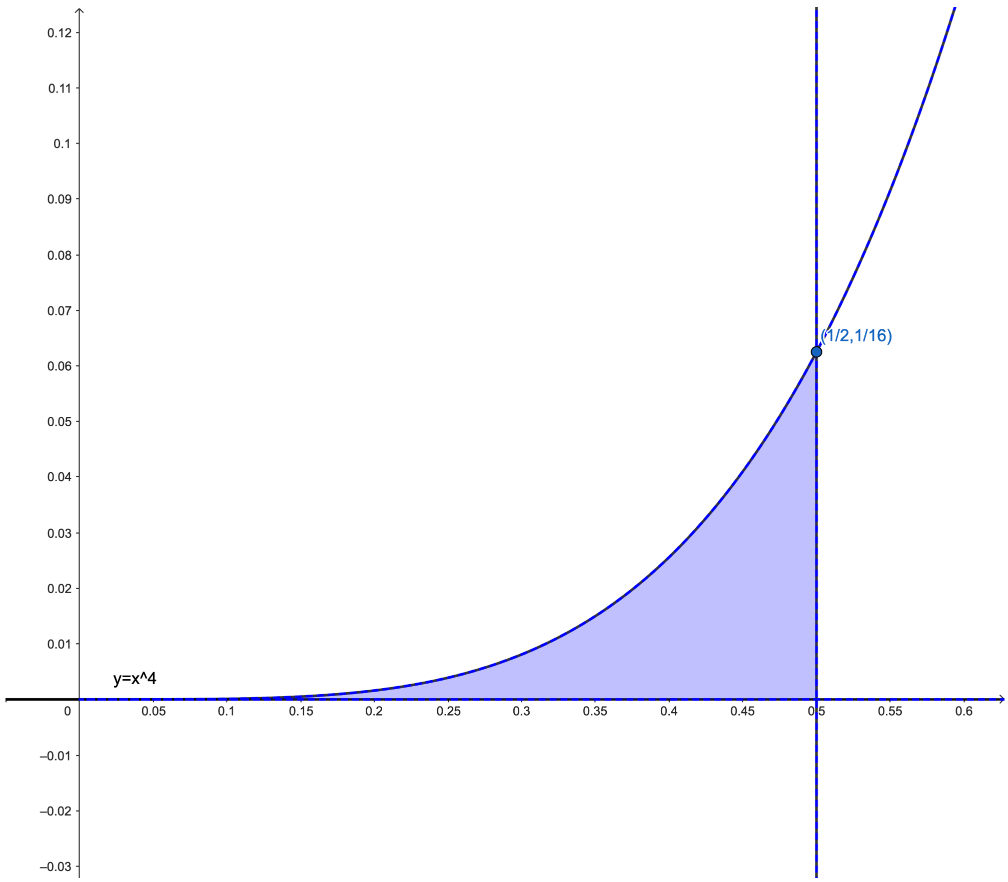
Solution.
Note that $\frac{d\tan^{-1}(x)}{dx} = \frac{1}{1+x^2}$.
Then,

$$\begin{aligned}
F(a) =& \int_0^a \int_0^x \frac{1}{1+t^2} dt dx \\
=& \int_0^a \int_t^a \frac{1}{1+t^2} dx dt \\
=& \int_0^a \frac{a-t}{1+t^2} dt \\
=&a \int_0^a \frac{1}{1+t^2} dt - \int_0^a \frac{t}{1+t^2} dt \\
=&a\left[\tan^{-1}(t)\right]_0^a - \left[\frac{1}{2}\ln\left(1+t^2\right)\right]_0^a \\
=&a\tan^{-1}(a) - \frac{1}{2}\ln\left(1+a^2\right)
\end{aligned}$$

Hence, $F(x) = x\tan^{-1}(x) - \frac{1}{2}\ln\left(1+x^2\right)$ is an antiderivative of $\tan^{-1}(x)$.
Check: $F'(x) = x\frac{1}{1+x^2} + \tan^{-1}(x) - \frac{1}{2}\frac{1}{1+x^2}2x = \tan^{-1}(x)$ □

Problem 4

Solution.



$$
\begin{aligned}
&\int_0^{1/16} \int_{y^{1/4}}^{1/2} \cos(16\pi x^5) dx dy \\
=&\int_0^{1/2} \int_0^{x^4} \cos(16\pi x^5) dy dx \\
=&\int_0^{1/2} \cos(16\pi x^5) x^4 dx \\
=&\int_0^{1/2} \frac{\cos(16\pi x^5)}{80\pi} d(16\pi x^5) \\
=&\left[\frac{\sin(16\pi x^5)}{80\pi}\right]_0^{1/2} \\
=&\frac{1}{80\pi}
\end{aligned}
$$

□

Problem 5

Solution.
Let $f(x,y) = \dfrac{\cos(x)-1}{x}, x \neq 0.$
Note that $\lim\limits_{x\to 0} \dfrac{\cos(x)-1}{x} = \lim\limits_{x\to 0} \dfrac{-\sin(x)}{1} = 0.$
We can set $f(x,y) = 0$ if $x = 0$ to obtain a continuous function. The continuity implies that we are allowed to apply Fubini's theorem.

$$\begin{aligned}
&\int_0^1 \int_y^1 \frac{\cos(x)-1}{x} dx dy \\
&= \int_0^1 \int_0^x \frac{\cos(x)-1}{x} dy dx \\
&= \int_0^1 \frac{\cos(x)-1}{x} x dx \\
&= \int_0^1 \cos(x) - 1 dx \\
&= [\sin(x) - x]_0^1 \\
&= \sin(1) - 1
\end{aligned}$$

□