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THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2020B Advanced Calculus II
Tutorial 1

Date: 15 January, 2025

1. Integrate the function $f(x, y) = x \ln y$ over the region $[-1, 2] \times [1, 2]$.

2. Find the volume below the elliptical paraboloid $z = 16 - x^2 - y^2$ and over $[0, 2] \times [0, 2]$.

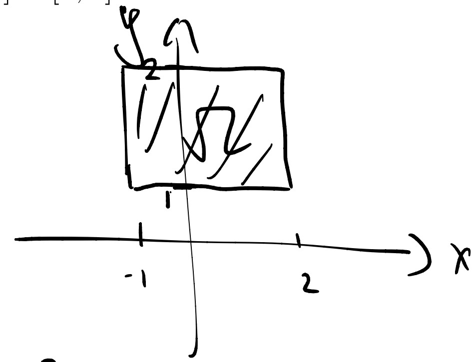
3. Evaluate the integral $\int\int_R xye^{xy^2} dxdy$ where $R = [0, 2] \times [0, 1]$.

4. Let $R = [0, 1] \times [0, 1]$ and let

$$f(x, y) = \begin{cases} 0, & x \text{ and } y \text{ are both rational} \\ 1, & \text{otherwise.} \end{cases}$$

Show that f is not integrable over R. *Hint: Consider an arbitrary partition P given by the subrectangles $R_1, R_2, \ldots, R_k, \ldots, R_n$ of R and consider two different classes of points $(x_k, y_k) \in R_k$ and $(x'_k, y'_k) \in R_k$ that give two distinct* [redacted]. values of the Riemann sum $L(P,f)$.

1. Integrate the function $f(x,y) = x\ln y$ over the region $[-1,2]\times[1,2]$.



let $\Omega = [-1,2]\times[1,2]$

$$\int_\Omega f = \int_{-1}^{2}\int_{1}^{2} f(x,y)\,dy\,dx$$

$$= \int_{-1}^{2}\int_{1}^{2} x\ln(y)\,dy\,dx = \int_{-1}^{2} x\left(y\ln(y)-y\right)\Big|_{y=1}^{y=2}\,dx$$

$$= \int_{-1}^{2} x\left(2\ln(2)-2-1\cancelto{0}{\ln(1)}+1\right)dx$$

$$= \left(2\ln(2)-1\right)\int_{-1}^{2} x\,dx = \left(2\ln(2)-1\right)\cdot\frac{1}{2}x^2\Big|_{x=-1}^{x=2}$$

$$= \left(2\ln(2)-1\right)\left(4-(-1)^2\right)$$

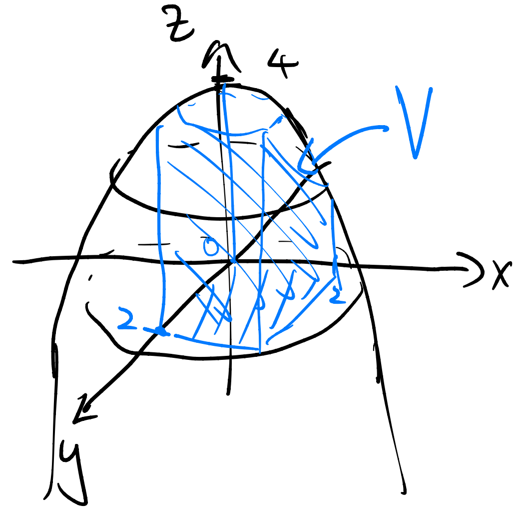
$$= \frac{6\ln(2)-3}{2} = \boxed{3\ln(2)-\frac{3}{2}}$$

Fubini's Thm $\Rightarrow$

$$\int_\Omega f = \int_{1}^{2}\int_{-1}^{2} x\ln(y)\,dx\,dy = \int_{1}^{2} \frac{1}{2}x^2\Big|_{x=-1}^{x=2}\cdot\ln(y)\,dy$$

$$= \int_{1}^{2} \frac{3}{2}\ln(y)\,dy = \cdots$$

2. Find the volume below the elliptical paraboloid $z = 16 - x^2 - y^2$ and over $[0,2]\times[0,2]$.



$$V = \int_\Omega z = \int_0^2 \int_0^2 (16 - x^2 - y^2)\,dy\,dx$$

$$= \int_0^2 \left(16y - x^2 y - \frac{1}{3}y^3\right)\Big|_{y=0}^{y=2} dx$$

$$= \int_0^2 \left(32 - 2x^2 - \frac{8}{3}\right) dx$$

$$= 32x\Big|_{x=0}^{x=2} - \frac{2}{3}x^3\Big|_{x=0}^{x=2} - \frac{8}{3}x\Big|_{x=0}^{x=2}$$

$$= \boxed{\frac{160}{3}}$$

3. Evaluate the integral $\int\int_R xye^{xy^2}dxdy$ where $R = [0,2] \times [0,1]$.

$$\int_0^2\int_0^1 xye^{xy^2}dydx$$ First integrate w.r.t. y:

$$u(y) = xy^2$$
$$du = 2xydy.$$
$$y=0 \Rightarrow u=0$$
$$y=1 \Rightarrow u=x.$$

$$\int_0^1 xye^{xy^2}dy = \int_0^x \frac{1}{2}e^u du$$
$$= \frac{1}{2}e^u\Big|_{u=0}^{u=x}$$
$$= \frac{1}{2}e^x - \frac{1}{2}.$$

So $$\int_0^2\int_0^1 xye^{xy^2}dydx = \int_0^2\left(\frac{1}{2}e^x - \frac{1}{2}\right)dx = \frac{1}{2}e^x\Big|_{x=0}^{x=2} - \frac{1}{2}x\Big|_{x=0}^{x=2}$$
$$= \frac{1}{2}e^2 - \frac{1}{2} - 1$$
$$= \boxed{\frac{1}{2}e^2 - \frac{3}{2}}$$

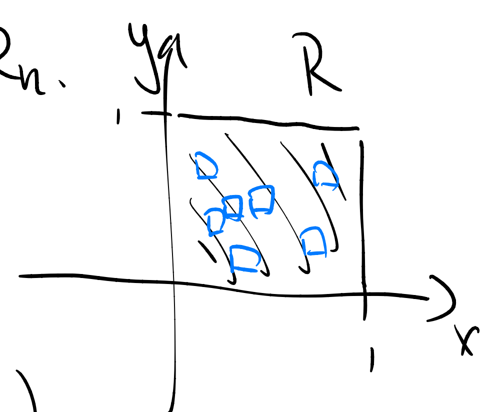
4. Let $R = [0,1] \times [0,1]$ and let

$$f(x,y) = \begin{cases} 0, & x \text{ and } y \text{ are both rational} \\ 1, & \text{otherwise.} \end{cases}$$

Show that f is not integrable over R. *Hint: Consider an arbitrary partition P given by the subrectangles $R_1, R_2, \ldots, R_k, \ldots, R_n$ of R and consider two different classes of points $(x_k, y_k) \in R_k$ and $(x'_k, y'_k) \in R_k$ that give two distinct* ~~*Riemann sums $L(P,f)$ and $L'(P,f)$*~~). values of the Riemann sum $L(P,f)$.

Pf: Let P be a partition of R.

$P = R_1, R_2, R_3, \ldots, R_k, \ldots, R_n$.



In each R_k, we can find points (x_k, y_k) where x_k, y_k are both rational.

and we can find points (x'_k, y'_k) where x'_k, y'_k are irrational (actually, only one needs to be irrational)

Then using the points (x_k, y_k), the Riemann sum:

$$L(P,f) = \sum_{k=1}^{n} \underbrace{f(x_k, y_k)}_{=0} \cdot \text{Area}(R_k) = \sum_{k=1}^{n} 0 \cdot \text{Area}(R_k)$$

$$= 0.$$

But using the points (x'_k, y'_k),

$$L(P,f) = \sum_{k=1}^{n} \underbrace{f(x'_k, y'_k)}_{=1} \cdot \text{Area}(R_k) = \sum_{k=1}^{n} \text{Area}(R_k)$$

$$= \text{Area}(R) = 1.$$

So $\lim_{\|P\| \to 0} L(P,f)$ DNE, so f is not integrable over R. //.