Tutorial 9 for MATH 2020A (2024 Fall)

1. Let C be the union of two curves $C_1 : \mathbf{r}(t) = (t, t^2 - t), 0 \le t \le 2$ and $C_2 : \mathbf{r}(t) = (2 - t, 2 - t), 0 \le t \le 2$. Let the vector field $\mathbf{F}(x, y) = (x^3y^2, \frac{1}{2}x^4y)$.

   (a)Draw the curve C on xy-plane and determine whether it belongs the types of regions to which Green's Theorem applies.

   (b)Find the **circulation** of $\mathbf{F}$ along C.

   (c)Find the **outward flux** of $\mathbf{F}$ across C.

   Solution: (b)0; (c)$\int_0^2 \int_{x^2-x}^{2-x} \left(3x^2y^2 + \frac{1}{2}x^4\right) \mathrm{d}y \, \mathrm{d}x$

2. Find the outward flux of the field

$$\mathbf{F}(x, y) = \left(3xy - \frac{x}{1 + y^2}\right) \mathbf{i} + (e^x + \arctan y) \mathbf{j}$$

   across the cardioid $r = a(1 + \cos\theta), a > 0$.

   Solution: 0

3. Evaluate the integral

$$\int_C \left(y^2 \, \mathrm{d}x + x^2 \, \mathrm{d}y\right),$$

   where C is the boundary of the triangle enclosed by the lines $x = 0$, $x + y = 1$, and $y = 0$, oriented in counterclockwise direction.

   Solution: 0

4. If a simple closed curve C in the plane and the region R it encloses satisfy the hypotheses of Green's Theorem,

   (a)Show that

$$Area(R) = \frac{1}{2} \int_C (x \, \mathrm{d}y - y \, \mathrm{d}x).$$

   (b)Use the formula above to calculate the area of the region enclosed by the astroid $\mathbf{r}(t) = (\cos^3 t)\mathbf{i} + (\sin^3 t)\mathbf{j}, 0 \le t \le 2\pi$.

Solution: (b)$\frac{3}{8}\pi$

5. Assuming that $f \in C^2(\mathbb{R}^2)$ and f satisfies the Laplace equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0,$$

show that

$$\int_C \left(\frac{\partial f}{\partial y} \, \mathrm{d}x - \frac{\partial f}{\partial x} \, \mathrm{d}y \right) = 0,$$

for all closed curves C to which Green's Theorem applies.

Solution: Directly apply Green's Theorem to the line integral.