Tutorial 12 for MATH 2020A (2024 Fall)

1. If $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is a differentiable vector field, we define the notation $\mathbf{F} \cdot \nabla$ to mean

$$M\frac{\partial}{\partial x} + N\frac{\partial}{\partial y} + P\frac{\partial}{\partial z}.$$

For differentiable vector fields $\mathbf{F}_1$ and $\mathbf{F}_2$, verify the following identities.

(a) $\nabla \times (\mathbf{F}_1 \times \mathbf{F}_2) = (\mathbf{F}_2 \cdot \nabla)\mathbf{F}_1 - (\mathbf{F}_1 \cdot \nabla)\mathbf{F}_2 + (\nabla \cdot \mathbf{F}_2)\mathbf{F}_1 - (\nabla \cdot \mathbf{F}_1)\mathbf{F}_2$.

(b) $\nabla(\mathbf{F}_1 \cdot \mathbf{F}_2) = (\mathbf{F}_1 \cdot \nabla)\mathbf{F}_2 + (\mathbf{F}_2 \cdot \nabla)\mathbf{F}_1 + \mathbf{F}_1 \times (\nabla \times \mathbf{F}_2) + \mathbf{F}_2 \times (\nabla \times \mathbf{F}_1)$.

Solution: Direct calculation

2. Recall that for a simple closed curve C in the plane, if the region R enclosed by this curve satisfy the hypotheses of Green's Theorem, then the area of R is given by

$$A(R) = \frac{1}{2}\int_C x\,\mathrm{d}y - y\,\mathrm{d}x.$$

Now consider the '8' curve $\Gamma : \mathbf{r}(t) = (\frac{1}{2}\sin 2t, \sin t), 0 \leq t \leq \pi$ (one loop).

(a) Sketch the curve Γ in xy-plane and label its orientation.

(b) Find the area of the region R enclosed by Γ.

Solution: (b)$\frac{2}{3}$

Feel free to ask questions during the remaining time of the tutorial session!