Tutorial 11 for MATH 2020A (2024 Fall)

1. Consider the vector field $\mathbf{F}(x, y, z) = (y, xz, x^2)$ and the oriented curve C : the boundary of the triangle cut from the plane $x + y + z = 1$ by the first octant, counterclockwise when viewed from above.

   (a) Sketch the curve C with its orientation, then determine the corresponding surface with a normal vector in Stokes' Theorem.

   (b) Calculate the circulation of $\mathbf{F}$ around C.

Solution: (b)$-\frac{5}{6}$

2. Let S be the cylinder $x^2 + y^2 = a^2, 0 \leq z \leq h$, together with its top, $x^2 + y^2 \leq a^2, z = h$. Let $\mathbf{F}(x, y, z) = (-y, x, x^2)$. Use Stokes' Theorem to find the flux of $\nabla \times \mathbf{F}$ through S in the direction away from the origin.

Solution: $2\pi a^2$

3. Let C be a simple closed smooth curve in the plane $2x + 2y + z = 2$, oriented in counterclockwise direction when viewed from the first octant.

   (a) Skecth one curve C satisfying the assumption above, then label its orientation.

   (b) Show that the line integral

$$\int_C 2y\,\mathrm{d}x + 3z\,\mathrm{d}y - x\,\mathrm{d}z$$

   depends only on the area of the region enclosed by C and not on the position or shape of C.

Solution: (b)$-6\cdot$Area(S), where S is the surface enclosed by C.

4. Let $D \subset \mathbb{R}^3$ be the region inside the solid cylinder $x^2 + y^2 \leq 4$ between the plane $z = 0$ and the paraboloid $z = x^2 + y^2$. Let $\mathbf{F}(x, y, z) = (y, xy, -z)$. Use the Divergence Theorem to find the outward flux of $\mathbf{F}$ across the boundary of the region D.

Solution: -8π

5. Suppose that f and g are scalar functions with continuous first- and second-order partial derivatives throughout a region D that is bounded by a closed piecewise smooth surface S. Show that

$$\iint_S f\nabla g \cdot \mathbf{n}\, \mathrm{d}\sigma = \iiint_D \left(f\nabla^2 g + \nabla f \cdot \nabla g\right)\, \mathrm{d}V.$$

(Recall that ∇^2 is the Laplacian operator defined by $\nabla^2 f = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$) This identity is known as **Green's first formula**.

Solution: Hint: Apply Divergence Theorem to the vector field $f\nabla g$, then invoke the following identity

$$\nabla \cdot (f\mathbf{F}) = \nabla f \cdot \mathbf{F} + f\nabla \cdot \mathbf{F}.$$