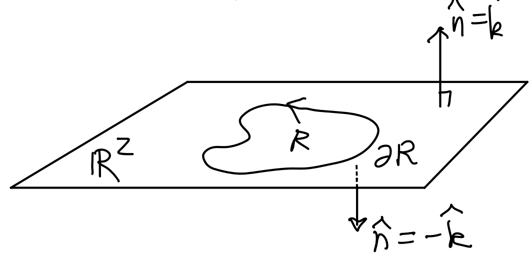
Remarks: (1) If we let $\vec{F} = M\hat{i} + N\hat{j} \longleftrightarrow \omega = M\,dx + N\,dy$

then $(\vec{\nabla}\times\vec{F})\underset{\hat{n}=\hat{k}}{\underset{\uparrow}{\cdot\hat{n}}}\,dA = (N_x - M_y)\underbrace{\hat{k}\cdot\hat{n}\,dA}_{dx\wedge dy} \longleftrightarrow d\omega$



Hence $\hat{k}\cdot\hat{n}\,dA = \begin{cases} dx\wedge dy & \text{if } \hat{n}=\hat{k} \\ dy\wedge dx & \text{if } \hat{n}=-\hat{k} \end{cases}$

(orientation of the "surface" domain)

(2) In fact, if $\vec{F} = M\hat{i} + N\hat{j} + L\hat{k} \longleftrightarrow \omega = M\,dx + N\,dy + L\,dz$

Then $d\omega = dM\wedge dx + dN\wedge dy + dL\wedge dz$

$$= (\cdots + M_y\,dy + M_z\,dz)\wedge dx + (N_x\,dx + \cdots + N_z\,dz)\wedge dy$$
$$+ (L_x\,dx + L_y\,dy + \cdots)\wedge dz$$
$$= (L_y - N_z)\,dy\wedge dz + (M_z - L_x)\,dz\wedge dx + (N_z - M_y)\,dx\wedge dy$$

Compare $\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ M & N & L \end{vmatrix} = (L_y - N_z)\hat{i} + (M_z - L_x)\hat{j} + (N_z - M_y)\hat{k}$

This suggest the correspondence (in $\mathbb{R}^3$):

$$\zeta_1\,dy\wedge dz + \zeta_2\,dz\wedge dx + \zeta_3\,dx\wedge dy \longleftrightarrow \vec{F} = \zeta_1\hat{i} + \zeta_2\hat{j} + \zeta_3\hat{k}$$

Then

$$\begin{array}{ccc} \vec{F} & \longleftrightarrow & \omega \\ \downarrow & & \downarrow \\ \vec{\nabla}\times\vec{F} & \longleftrightarrow & d\omega \end{array}$$

(1-form correspondence)

(2-form correspondence)

(3) Similarly

$$\begin{array}{ccc} & f & \\ \swarrow & & \searrow \\ \vec{\nabla}f & \longleftrightarrow & df \end{array}$$

(1-form correspondence)

And, if we make the following correspondence

$$f\, dx\wedge dy\wedge dz \longleftrightarrow f$$

then

$$\begin{array}{ccc} \zeta & \longleftrightarrow & \vec{F} \\ \downarrow & & \downarrow \\ d\zeta & \longleftrightarrow & \vec{\nabla}\cdot\vec{F} \end{array}$$

(2-form correspondence)

(3-form & 0-form correspondence)

See next example:

eg2 : $\zeta = \zeta_1\, dy\wedge dz + \zeta_2\, dz\wedge dx + \zeta_3\, dx\wedge dy$

Then $d\zeta = d\zeta_1\wedge dy\wedge dz + d\zeta_2\wedge dz\wedge dx + d\zeta_3\wedge dx\wedge dy$

$$= \left(\frac{\partial\zeta_1}{\partial x}dx + \cdots\right)\wedge dy\wedge dz + \left(\cdots + \frac{\partial\zeta_2}{\partial y}dy + \cdots\right)\wedge dz\wedge dx$$

$$+ \left(\cdots + \frac{\partial\zeta_3}{\partial z}dz\right)\wedge dx\wedge dy$$

$$= \left(\frac{\partial\zeta_1}{\partial x} + \frac{\partial\zeta_2}{\partial y} + \frac{\partial\zeta_3}{\partial z}\right) dx\wedge dy\wedge dz$$

$$= (\vec{\nabla}\cdot\vec{F})\, \underline{dx\wedge dy\wedge dz}$$

+ve oriented volume element

where $\vec{F} = \zeta_1\hat{i} + \zeta_2\hat{j} + \zeta_3\hat{k}$

Hence the divergence thm can be written as

$$\iiint_D d\zeta = \iiint_D \left(\frac{\partial \zeta_1}{\partial x} + \frac{\partial \zeta_2}{\partial y} + \frac{\partial \zeta_3}{\partial z}\right) dx \wedge dy \wedge dz$$

$$= \iiint_D (\vec{\nabla} \cdot \vec{F})\, dV = \iint_{\partial D} \vec{F} \cdot \hat{n}\, d\sigma$$

$\uparrow$ outward unit normal on ∂D

To see the relation between $\vec{F} \cdot \hat{n}\, d\sigma$ and ζ,

we parametrize ∂D:

$$\vec{r}(u,v) = x(u,v)\,\hat{i} + y(u,v)\,\hat{j} + z(u,v)\,\hat{k}$$

$$\begin{cases} \vec{r}_u = x_u \hat{i} + y_u \hat{j} + z_u \hat{k} \\ \vec{r}_v = x_v \hat{i} + y_v \hat{j} + z_v \hat{k} \end{cases}$$

$$\Rightarrow \vec{r}_u \times \vec{r}_v = \begin{vmatrix} y_u & y_v \\ z_u & z_v \end{vmatrix} \hat{i} + \begin{vmatrix} z_u & z_v \\ x_u & x_v \end{vmatrix} \hat{j} + \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \hat{k}$$

$$= \frac{\partial(y,z)}{\partial(u,v)} \hat{i} + \frac{\partial(z,x)}{\partial(u,v)} \hat{j} + \frac{\partial(x,y)}{\partial(u,v)} \hat{k}$$

If $\vec{r}_u \times \vec{r}_v$ is <u>outward</u>, then

$$\hat{n} = \frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|} \quad \& \quad d\sigma = \|\vec{r}_u \times \vec{r}_v\|\, du\,dv$$

$$= \|\vec{r}_u \times \vec{r}_v\|\, du \wedge dv$$

(if orientation correct)

Then $\vec{F} \cdot \hat{n}\, d\sigma = \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\, du \wedge dv$

$$= \left(\zeta_1 \frac{\partial(y,z)}{\partial(u,v)} + \zeta_2 \frac{\partial(z,x)}{\partial(u,v)} + \zeta_3 \frac{\partial(x,y)}{\partial(u,v)}\right) du \wedge dv$$

$$= \zeta_1\, dy \wedge dz + \zeta_2\, dz \wedge dx + \zeta_3\, dx \wedge dy = \zeta$$

Hence divergence thm is

$$\boxed{\iiint_D d\xi = \iint_{\partial D} \xi}$$

#

<u>eg3</u> <u>Stokes' Thm</u>

$$\vec{F} = M\hat{i} + N\hat{j} + L\hat{k} \quad \longleftrightarrow \quad \omega = M\,dx + N\,dy + L\,dz$$

Then $\vec{\nabla}\times\vec{F} \leftrightarrow$

$$d\omega = (L_y - N_z)\,dy\wedge dz + (M_z - L_x)\,dz\wedge dx + (N_x - N_y)\,dx\wedge dy$$

As in eg2, $\hat{n} = \dfrac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$, $d\sigma = \|\vec{r}_u \times \vec{r}_v\|\,du\wedge dv$

(with correct orientation)

$$(\vec{\nabla}\times\vec{F})\cdot\hat{n}\,d\sigma = (\vec{\nabla}\times\vec{F})\cdot(\vec{r}_u\times\vec{r}_v)\,du\wedge dv$$

$$= \left[(L_y - N_z)\frac{\partial(y,z)}{\partial(u,v)} + (M_z - L_x)\frac{\partial(z,x)}{\partial(u,v)} + (N_x - M_y)\frac{\partial(x,y)}{\partial(u,v)}\right] du\wedge dv$$

$$= (L_y - N_z)\,dy\wedge dz + (M_z - L_x)\,dz\wedge dx + (N_x - M_y)\,dx\wedge dy$$

$$= d\omega$$

$\therefore$ Stokes' Thm

$$\oint_{\partial S}\vec{F}\cdot d\vec{r} \longrightarrow \boxed{\oint_{\partial S}\omega = \iint_S d\omega} \longleftarrow \iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,d\sigma$$

#

Generalization to manifold of n-dimension with boundary

(Skipped)

- M = n dim'l Manifold (oriented)
- ∂M = boundary (oriented with "induced" orientation)
- ω = $(n-1)$-form on M (smooth)

Then $$\boxed{\int_M d\omega = \int_{\partial M} \omega}$$

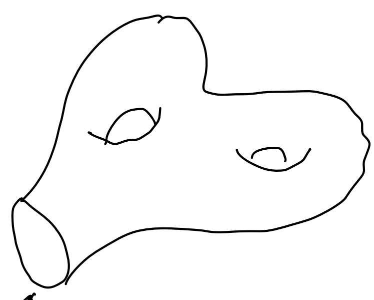
$\int_M$: n-dim'l integral

$\int_{\partial M}$: $(n-1)$-dim'l integral

Note: ∂M is always closed, i.e. no boundary.

$$\therefore \boxed{\partial(\partial M) = \partial^2 M = 0}$$

boundary has no boundary



∂S is a closed curve

Hence if $\omega = d\eta$, for some $(n-2)$-form η,

then $\int_M d(d\eta) = \int_M d\omega = \int_{\partial M} \omega$

$= \int_{\partial M} d\eta = \int_{\partial(\partial M)} \eta = 0$ (for any η.)

This suggests $\boxed{d^2\eta = 0}$, $\forall$ differential form η

<u>Pf</u> (of 0-form and 1-form in $\mathbb{R}^3$)

- <u>0-form</u> $\eta = f$ a (smooth) function

Then

$$\begin{array}{ccccc} & & f & & \\ & \swarrow & & \searrow & \\ df & \longleftrightarrow & & \vec{\nabla} f & \text{(1-form)} \\ \downarrow & & & \downarrow & \\ d^2 f & \longleftrightarrow & & \boxed{\begin{array}{c} \vec{\nabla}\times(\vec{\nabla} f) \\ \| \quad \forall f \\ 0 \end{array}} & \text{(2-form)} \end{array}$$

$\therefore\ d^2 f = 0$

- 1-form

$$\begin{array}{ccccc} \eta & \longleftrightarrow & \vec{F} & \\ \downarrow & & \downarrow & \\ d\eta & \longleftrightarrow & \vec{\nabla}\times\vec{F} & \text{(2-form)} \\ \downarrow & & \downarrow & \\ d^2\eta & \longleftrightarrow & \boxed{\begin{array}{c} \vec{\nabla}\cdot(\vec{\nabla}\times\vec{F}) \\ \| \quad \forall \vec{F} \\ 0 \end{array}} & \text{(3-form)} \end{array}$$

$\Rightarrow\ d^2\eta = 0$

Remark: $d^2(\text{2-form}) = \text{4-form} = 0$ in $\mathbb{R}^3$

$d^2(\text{3-form}) = \text{5-form} = 0$ in $\mathbb{R}^3$

$\therefore$ $\forall$ diff. form η, $d^2\eta = 0$.

eg: Let $\omega = \dfrac{-y}{x^2+y^2}\,dx + \dfrac{x}{x^2+y^2}\,dy$

check: $d\omega = 0$ ($d\omega \Leftrightarrow \vec{\nabla}\times\vec{F} = \vec{0}$)

But $\omega \neq df$ for any smooth function on $\mathbb{R}^2\setminus\{(0,0)\}$

$(\Longleftrightarrow \vec{F} \neq \vec{\nabla} f)$

(since $\omega = d\theta$ and θ is not defined on $\mathbb{R}^2\setminus\{(0,0)\}$)

Hence $d\omega = 0 \not\Rightarrow \omega = d\eta$ in general

$(\Longleftarrow)$

Note: Thm 10 can be written as:

$$\boxed{\begin{array}{l} \Omega \subset \mathbb{R}^2 \text{ \underline{simply-connected}, then (for all 1-form } \omega) \\ d\omega = 0 \Leftrightarrow \omega = df \text{ for some smooth function } f \text{ on } \Omega \end{array}}$$

(End of Term)

Review

Double integrals

- Riemann sum, integrability, Fubini's Thm,
- Polar coordinates, improper integrals
- Applications: area, average, etc.

Triple integrals

- Riemann sum, integrability, Fubini's Thm,
- cylindrical & spherical coordinates, improper integrals
- Applications: volume, average, etc

Change of Variables

- Chain Rule, Jacobian (determinant)

mid-term

Vector Analysis

- Vector fields, gradient of a function ($\vec{\nabla} f$)
- line integral of functions, arc-length
- line integral of vector fields: flow & flux
- simple-closed curves, orientation of curves
- Conservative vector fields (Thms 8, 9 & 10)
- Simply-connected domains
- Curl & Div ($\vec{\nabla} \times \vec{F}$, $\vec{\nabla} \cdot \vec{F}$)

- Surface integrals, area elements, orientation,
- Surface integrals of vector fields (flux)
- Green's, Stokes' & Divergence Thm
- Differential Forms

Final Exam   Dec 12 (Thu) 9:30 – 11:30 am SRRSH (Stage)

Coverage: • All material in lecture notes, tutorial notes, textbook (Ch 14 & 15) & homework assignments,

- except differential forms
- emphasis on those material not included in Midterm.
- 5 questions, answer all. Some are unfamiliar/difficult questions as required by the grade descriptor of A range,

(Note: Textbook & assignments contain only basic theory and basic questions.)

(End)