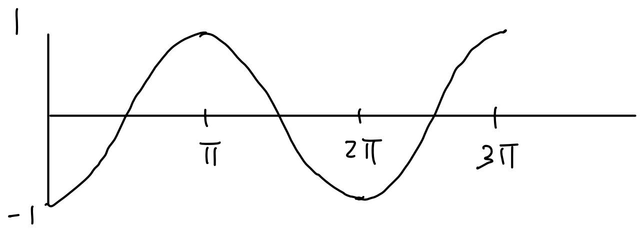
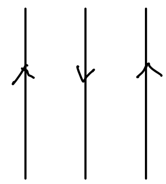
## supplementary egs

(1) $\begin{cases} x = u & , u \in [0,1] \\ y = -\cos\theta, & \theta \in [0, 3\pi] \end{cases}$ onto, but not one-to-one

$$[0,1] \times [0, 3\pi] \longrightarrow [0,1] \times [-1,1]$$



covers $[-1,1]$ 3 times:
with 2 positive direction
1 negative direction

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \sin\theta \end{pmatrix} \Rightarrow \frac{\partial(x,y)}{\partial(u,\theta)} = \sin\theta$$

- $$\iint_{[0,1]\times[-1,1]} dA = \iint_{[0,1]\times[-1,1]} dx\,dy = \text{Area}([0,1]\times[-1,1]) = 2$$

- $$\iint_{[0,1]\times[0,3\pi]} \left|\frac{\partial(x,y)}{\partial(u,\theta)}\right| du\,d\theta = \int_0^{3\pi}\int_0^1 |\sin\theta|\,du\,d\theta$$
$$= \int_0^{3\pi} |\sin\theta|\,d\theta$$
$$= \int_0^{\pi} \sin\theta\,d\theta + \int_{\pi}^{2\pi} (-\sin\theta)\,d\theta + \int_{2\pi}^{3\pi} \sin\theta\,d\theta$$
$$= 6 \quad (3 \times |\pm \text{Area}|)$$
$$\neq \text{Area}([0,1]\times[-1,1])$$

- $$\iint_{[0,1]\times[0,3\pi]} \frac{\partial(x,y)}{\partial(u,\theta)}\,du\,d\theta = \int_0^{3\pi}\int_0^1 \sin\theta\,du\,d\theta = \int_0^{3\pi} \sin\theta\,d\theta$$
$$= 2 \quad (2 \times (+\text{Area}) + 1 \times (-\text{Area}))$$

In general, if $(u,v) \mapsto (x,y)$ is not <u>1-1</u> onto,

$$\iint f(x,y)\,dxdy \neq \iint f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv,$$

but may still have (under suitable conditions)

$$\iint f(x,y)\,dx\wedge dy = \iint f(x(u,v), y(u,v)) \frac{\partial(x,y)}{\partial(u,v)}\, du\wedge dv.$$

(2) $\begin{cases} x = v \\ y = u \end{cases}$ $(u,v) \in [0,1]\times[0,1]$ (one-to-one & onto)

$[0,1]\times[0,1] \to [0,1]\times[0,1]$

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = -1$$

- $\displaystyle\iint_{[0,1]\times[0,1]} dA = \iint_{[0,1]\times[0,1]} dxdy = \text{Area}([0,1]\times[0,1]) = 1$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad ||$ change of variables formula is verified.

- $\displaystyle\iint_{[0,1]\times[0,1]} \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv = \iint_{[0,1]\times[0,1]} |-1|\,dudv = 1$

- $\displaystyle\iint_{[0,1]\times[0,1]} \frac{\partial(x,y)}{\partial(u,v)}\, dudv = \iint_{[0,1]\times[0,1]} (-1)\,dudv = -1$

New formula

$$\iint_{[0,1]\times[0,1]} dx\wedge dy = \iint_{[0,1]\times[0,1]} \frac{\partial(x,y)}{\partial(u,v)}\, du\wedge dv = \iint_{[0,1]\times[0,1]} -du\wedge dv = \iint_{[0,1]\times[0,1]} dv\wedge du$$

\#

<u>Reason</u> : (1) $dxdy$ or $dydx$ in double integrals represent the same area element, and

$dxdydz$, $dydxdz$, ... etc in triple integrals represent the same volume element.

No orientation (in the definition)

(Perhaps also represent the "order" of iterated integrals using Fubini's Thm.)

(2) However, $dx\wedge dy$ and $dy\wedge dx$ can be used to represent the <u>area element</u> and the <u>orientation</u> of the "surfaces" in one notation. (see later remark)

Similarly, for $dx\wedge dy\wedge dz$ & etc.

## Exterior differentiation "d" on a form "$\omega$".

| | |
|---|---|
| (0-form) $f$ | $df$ (1-form) |
| (1-form) $\omega = \omega_1 dx + \omega_2 dy + \omega_3 dz$ | $d\omega = d\omega_1 \wedge dx + d\omega_2 \wedge dy + d\omega_3 \wedge dz$ (2-form) |
| $\zeta = \zeta_1 dy \wedge dz + \zeta_2 dz \wedge dx + \zeta_3 dx \wedge dy$ (2-form) | $d\zeta = d\zeta_1 \wedge dy \wedge dz + d\zeta_2 \wedge dz \wedge dx + d\zeta_3 \wedge dx \wedge dy$ (3-form) |
| (3-form) $f\, dx \wedge dy \wedge dz$ | $df \wedge dx \wedge dy\, dz = 0$ (4-form) in $\mathbb{R}^3$ |

eg 0 : $\quad d(dx) = d(dy) = d(dz) = 0$

eg 1 (in $\mathbb{R}^2$) $\quad \omega = M\,dx + N\,dy \qquad (M = M(x,y),\ N = N(x,y))$

then
$$d\omega = dM \wedge dx + dN \wedge dy$$
$$= (M_x dx + M_y dy) \wedge dx + (N_x dx + N_y dy) \wedge dy$$
$$= (N_x - M_y)\, dx \wedge dy$$

(+ve) oriented area element

In this notation, Green's Thm

$$\oint_{C = \partial R} M\,dx + N\,dy = \iint_R (N_x - M_y)\, dA \qquad (C \text{ is anti-clockwise})$$

can be written as
$$\boxed{\oint_{\partial R} \omega = \iint_R d\omega}$$