Proof of Divergence Thm

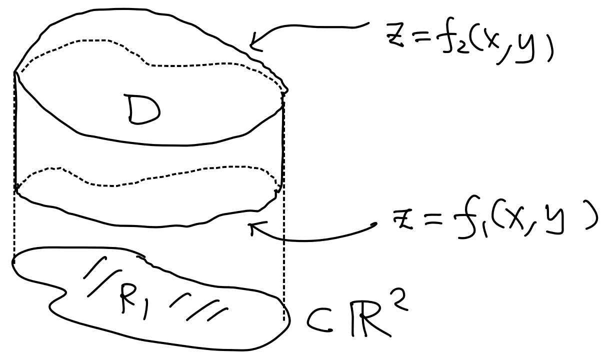
Same as Green's Thm, we'll prove only the case of special domain D which is of type I, II & III.

$$D = \{(x,y,z) \in \mathbb{R}^3 : (x,y) \in R_1,\ f_1(x,y) \le z \le f_2(x,y)\} \quad \text{(type I)}$$

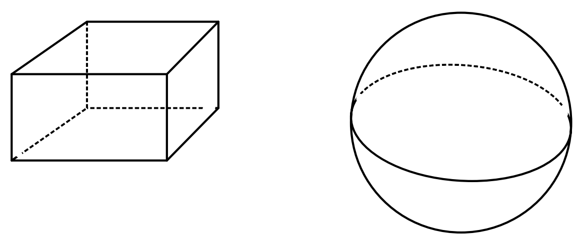
$$= \{(x,y,z) \in \mathbb{R}^3 : (y,z) \in R_2,\ g_1(y,z) \le x \le g_2(y,z)\} \quad \text{(type II)}$$

$$= \{(x,y,z) \in \mathbb{R}^3 : (x,z) \in R_3,\ h_1(x,z) \le y \le h_2(x,z)\} \quad \text{(type III)}$$

egs (i) Type I domain



(ii) Special domains:



And also as in the proof of Green's Thm for

$$\vec{F} = M\hat{i} + N\hat{j} + L\hat{k}$$

we'll prove 3 equalities in the following which combine to give the divergence thm:

$$\begin{cases} \iint_S M\hat{i}\cdot\hat{n}\,d\sigma = \iiint_D \frac{\partial M}{\partial x}\,dV & (\text{by type II}) \\ \iint_S N\hat{j}\cdot\hat{n}\,d\sigma = \iiint_D \frac{\partial N}{\partial y}\,dV & (\text{by type III}) \\ \iint_S L\hat{k}\cdot\hat{n}\,d\sigma = \iiint_D \frac{\partial L}{\partial z}\,dV & (\text{by type I}) \end{cases}$$

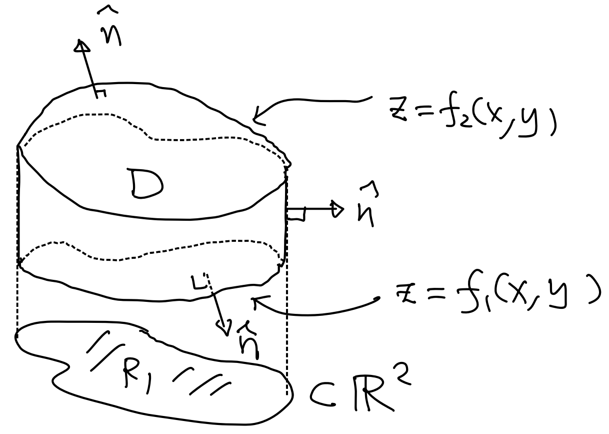
The proof are similar, we'll prove only the last one:

$$\iint_S L\hat{k}\cdot\hat{n}\,d\sigma = \iiint_D \frac{\partial L}{\partial z}\,dV$$

By Fubini's Thm

$$\text{RHS} = \iiint_D \frac{\partial L}{\partial z}\,dV = \iint_{R_1}\left[\int_{f_1(x,y)}^{f_2(x,y)} \frac{\partial L}{\partial z}\,dz\right]dx\,dy \quad (\text{type I})$$

$$= \iint_{R_1}\left[L(x,y,f_2(x,y)) - L(x,y,f_1(x,y))\right]dx\,dy$$



For the LHS, we note that by definition of type I domain, the boundary surface $S(=\partial D)$ of D can be written as

$$S = S_1 \cup S_2 \cup S_3$$

where S_1 = graph of $f_1 = \{z = f_1(x,y)\} = \{(x,y,f_1(x,y))\}$

S_2 = graph of $f_2 = \{z = f_2(x,y)\} = \{(x,y,f_2(x,y))\}$

S_3 = vertical surface (could be empty) between S_1 & S_2

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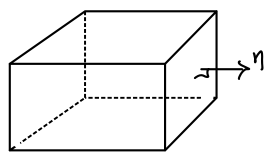
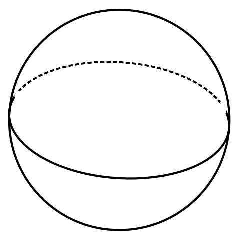


S_3 = union of 4 vertical rectangles



$S_3 = \phi$

)

Hence $\text{LHS} = \iint_S L\hat{k}\cdot\hat{n}\,d\sigma = \iint_{S_1} L\hat{k}\cdot\hat{n}\,d\sigma + \iint_{S_2} L\hat{k}\cdot\hat{n}\,d\sigma$

$$+ \iint_{S_3} \cancel{L\hat{k}\cdot\hat{n}\,d\sigma} \to 0$$

(Since $\hat{n}$ of a vertical surface is horizontal, $\hat{k}\cdot\hat{n} = 0$)

Now on the upper surface $S_2 = \{(x,y,f_2(x,y))\}$

the outward unit normal $\hat{n}$ is upward (i.e. $\hat{n}\cdot\hat{k} \geq 0$)

Note that the parametrization

$$(x,y) \mapsto \vec{r}(x,y) = (x,y,f_2(x,y))$$

has $\begin{cases} \vec{r}_x = \hat{i} + \dfrac{\partial f_2}{\partial x}\hat{k} \\ \vec{r}_y = \hat{j} + \dfrac{\partial f_2}{\partial y}\hat{k} \end{cases}$

and $\vec{r}_x \times \vec{r}_y = -\dfrac{\partial f_2}{\partial x}\hat{i} - \dfrac{\partial f_2}{\partial y}\hat{j} + \hat{k}$ ⟵ $\vec{r}_x \times \vec{r}_y$ is outward

$\Rightarrow$ $\hat{n} = \dfrac{\vec{r}_x \times \vec{r}_y}{\|\vec{r}_x \times \vec{r}_y\|}$ is the upward unit normal

& $\hat{k}\cdot\hat{n} = \dfrac{1}{\|\vec{r}_x \times \vec{r}_y\|}$

Therefore $\displaystyle\iint_{S_2} L\hat{k}\cdot\hat{n}\, d\sigma = \iint_{R_1} L(x,y,f_2(x,y)) \frac{1}{\cancel{\|\vec{r}_x \times \vec{r}_y\|}} \cancel{\|\vec{r}_x \times \vec{r}_y\|}\, dx\,dy$

$\displaystyle = \iint_{R_1} L(x,y,f_2(x,y))\, dx\,dy$

Similarly, note that the outward normal on S_1 (lower surface) is downward (i.e. $\hat{n}\cdot\hat{k} \leq 0$), we have

$\hat{n} = -\dfrac{\vec{r}_x \times \vec{r}_y}{\|\vec{r}_x \times \vec{r}_y\|}$, where $\vec{r}(x,y) = (x, y, f_1(x,y))$

$\Rightarrow$ $\hat{k}\cdot\hat{n} = -\dfrac{1}{\|\vec{r}_x \times \vec{r}_y\|}$

Hence $\displaystyle\iint_{S_1} L\hat{k}\cdot\hat{n}\, d\sigma = -\iint_{R_1} L(x,y,f_1(x,y))\, dx\,dy$

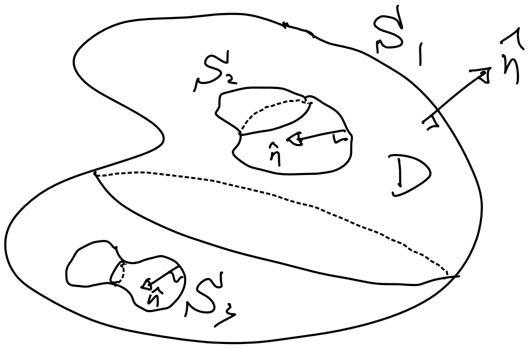
$\displaystyle\therefore \iint_{S} L\hat{k}\cdot\hat{n}\, d\sigma = \iint_{R_1} L(x,y,f_2(x,y))\, dx\,dy - \iint_{R_1} L(x,y,f_1(x,y))\, dx\,dy$

$$= \iint_{R_1} \left[L(x,y,f_2(x,y)) - L(x,y,f_1(x,y)) \right] dxdy$$

$$\equiv \iiint_D \frac{\partial L}{\partial z}\, dV$$

This completes the proof of the Divergence Thm. #

<u>Note</u>: Similar to Green's Thm, the Divergence Thm also holds for solid region with finitely many holes insides:



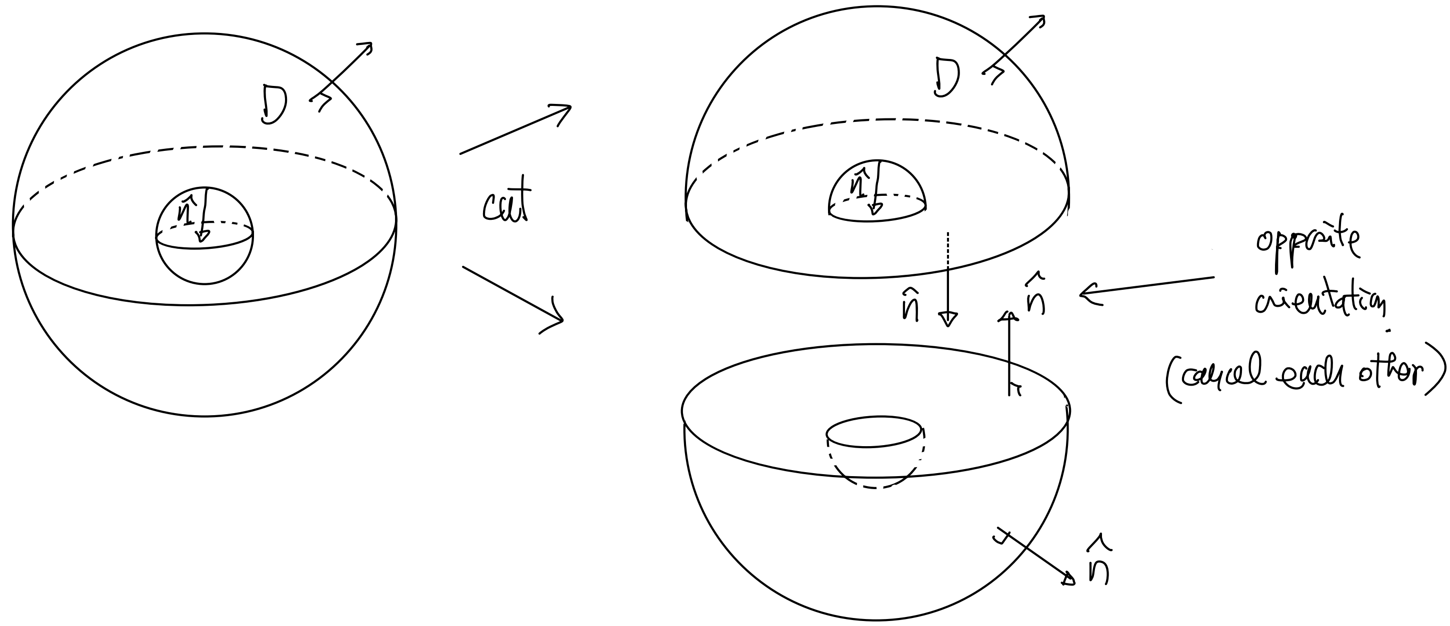
D = solid region inside S_1

but outside of S_2 and S_3

$$\boxed{\iiint_D \vec{\nabla}\cdot\vec{F}\, dV = \sum_{i=1}^{n} \iint_{S_i} \vec{F}\cdot\hat{n}\, d\sigma}$$

for $\hat{n}$ = <u>outward</u> normal <u>with respect to D</u>.

eg (idea of proof of this kind of surface :)



<u>Note</u>: Physical meaning of $\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F}$ in $\mathbb{R}^3$

= <u>flux density</u> (by the divergence thm)

Unified treatment of Green's, Stokes', and Divergence Theorems

Stokes' Thm in notations of differential forms (in $\mathbb{R}^3$)

Working definition of differential forms

(1) A differential 1-form (or simply 1-form)

is a linear combination of the symbols dx, dy & dz:

$$\boxed{\omega = \omega_1\, dx + \omega_2\, dy + \omega_3\, dz}$$

with coefficients $\omega_1, \omega_2, \omega_3$ functions on $\mathbb{R}^3$.

eg: Total differential of a function f on $\mathbb{R}^3$:

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \quad \text{is a 1-form.}$$

(2) Wedge product: Let "$\wedge$" be an operation such that

$$\begin{cases} dx\wedge dx = dy\wedge dy = dz\wedge dz = 0 \\ dx\wedge dy = -dy\wedge dx \\ dy\wedge dz = -dz\wedge dy \\ dz\wedge dx = -dx\wedge dz \end{cases}$$

and satisfies other usual rules in arithmetic.

i.e. If $\omega = \omega_1 dx + \omega_2 dy + \omega_3 dz$

$\eta = \eta_1 dx + \eta_2 dy + \eta_3 dz$

then we have

$$\omega\wedge\eta = (\omega_1 dx + \omega_2 dy + \omega_3 dz)\wedge(\eta_1 dx + \eta_2 dy + \eta_3 dz)$$

$$= (\omega_1 dx)\wedge(\eta_1 dx) + (\omega_2 dy)\wedge(\eta_1 dx) + (\omega_3 dz)\wedge(\eta_1 dx)$$

$$+ (\omega_1 dx)\wedge(\eta_2 dy) + (\omega_2 dy)\wedge(\eta_2 dy) + (\omega_3 dz)\wedge(\eta_2 dy)$$

$$+ (\omega_1 dx)\wedge(\eta_3 dz) + (\omega_2 dy)\wedge(\eta_3 dz) + (\omega_3 dz)\wedge(\eta_3 dz)$$

$$= \omega_1\eta_1 \cancel{dx\wedge dx} + \omega_2\eta_1 dy\wedge dx + \omega_3\eta_1 dz\wedge dx$$
$$+ \omega_1\eta_2 dx\wedge dy + \omega_2\eta_2 \cancel{dy\wedge dy} + \omega_3\eta_2 dz\wedge dy$$
$$+ \omega_1\eta_3 dx\wedge dz + \omega_2\eta_3 dy\wedge dz + \omega_3\eta_3 \cancel{dz\wedge dz}$$

$$\therefore \quad \boxed{\begin{aligned}\omega\wedge\eta = (\omega_2\eta_3 - \omega_3\eta_2)\, dy\wedge dz \\ + (\omega_3\eta_1 - \omega_1\eta_3)\, dz\wedge dx \\ + (\omega_1\eta_2 - \omega_2\eta_1)\, dx\wedge dy\end{aligned}}$$

- Linear combinations of $dy\wedge dz$, $dz\wedge dx$ & $dx\wedge dy$ are called <u>differential 2-forms</u> (on $\mathbb{R}^3$)

$$\boxed{\zeta = \zeta_1\, dy\wedge dz + \zeta_2\, dz\wedge dx + \zeta_3\, dx\wedge dy}$$

Similarly, if ω is a 1-form and

ζ is a 2-form

then we can define $\omega\wedge\zeta$

eg: If $\omega = dx$, $\xi = dy \wedge dz$

then $\omega \wedge \xi = dx \wedge dy \wedge dz$

Note that we insist on the anti-commutativity of wedge product,

we have

$$
\begin{aligned}
dx \wedge dy \wedge dz &= -\, dy \wedge dx \wedge dz \\
&= \quad dy \wedge dz \wedge dx \\
&= -\, dz \wedge dy \wedge dx \\
&= \quad dz \wedge dx \wedge dy \\
&= -\, dx \wedge dz \wedge dy
\end{aligned}
$$

And $dx \wedge dx \wedge dy = \cdots = 0$ whenever one of the dx, dy, dz is repeated.

Hence, as $\dim \mathbb{R}^3 = 3$, all "linear combinations" of "3-forms" are just $f\, dx \wedge dy \wedge dz$

which is called a differential 3-form (also called a volume form if $f > 0$)

Note: It is convenient to call smooth functions f the differential 0-form.

<u>Summary (on $\mathbb{R}^3$)</u>

0 - form	=	f
1 - form	=	$\omega_1 dx + \omega_2 dy + \omega_3 dz$
2 - form	=	$\xi_1 dy\wedge dz + \xi_2 dz\wedge dx + \xi_3 dx\wedge dy$
3 - form	=	$g\, dx\wedge dy\wedge dz$

where, f, g, ω_i, ξ_i are (smooth) functions

<u>Note</u>: One can certainly define k-form for any $k\geq 0$. But in $\mathbb{R}^3$, k-forms are zero for $k>3$:

$$dx^i \wedge dx \wedge dy \wedge dz = 0, \quad \text{where } dx^i = dx, dy, \text{ or } dz.$$

Change of Variables Formula: ($\mathbb{R}^2$)

$$\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$$

$$\Rightarrow \begin{cases} dx = x_u\,du + x_v\,dv \\ dy = y_u\,du + y_v\,dv \end{cases}$$

$$\Rightarrow dx \wedge dy = (x_u\,du + x_v\,dv) \wedge (y_u\,du + y_v\,dv)$$

$$= (x_u y_v - x_v y_u)\, du \wedge dv$$

$$= \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} du \wedge dv$$

$$= \frac{\partial(x,y)}{\partial(u,v)}\, du \wedge dv \qquad \text{(no absolute value)}$$

$$\boxed{dx \wedge dy = \frac{\partial(x,y)}{\partial(u,v)}\, du \wedge dv}$$

Jacobian determinant.

Hence naturally

$$\boxed{\iint f(x,y)\, dx \wedge dy = \iint f(x(u,v), y(u,v))\, \frac{\partial(x,y)}{\partial(u,v)}\, du \wedge dv}$$

Compare with

$$\boxed{\iint f(x,y)\, dx\, dy = \iint f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv}$$

Similarly for $\begin{cases} x = x(u,v,w) \\ y = y(u,v,w) \\ z = z(u,v,w) \end{cases}$

Then $$\boxed{dx\wedge dy\wedge dz = \frac{\partial(x,y,z)}{\partial(u,v,w)}\, du\wedge dv\wedge dw}$$ (no absolute value)

Pf: $$dx\wedge dy\wedge dz = (x_u du + x_v dv + x_w dw)\wedge(y_u du + y_v dv + y_w dw)\wedge(z_u du + z_v dv + z_w dw)$$

$$\begin{aligned} &= (x_u y_v z_w - x_u y_w z_v)\, du\wedge dv\wedge dw \\ &\quad + (-x_v y_u z_w + x_v y_w z_u)\, du\wedge dv\wedge dw \\ &\quad + (x_w y_u z_v - x_w y_v z_u)\, du\wedge dv\wedge dw \quad \text{(Ex!)} \\ &= [x_u(y_v z_w - y_w z_v) - x_v(y_u z_w - y_w z_u) + x_w(y_u z_v - y_v z_u)]\, du\wedge dv\wedge dw \\ &= \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix} du\wedge dv\wedge dw \end{aligned}$$ ✗

- "Oriented" change of variables formula
- "$dx\wedge dy$" oriented area element
- "$dx\wedge dy\wedge dz$" oriented volume element.

(see later remark)