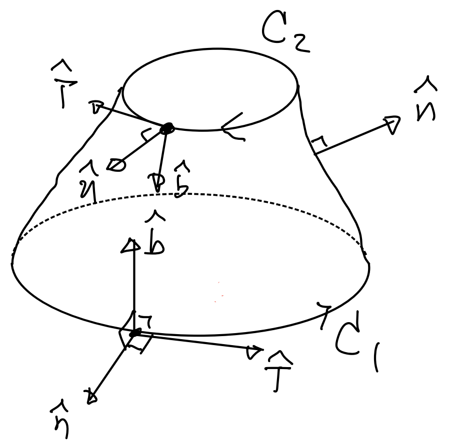
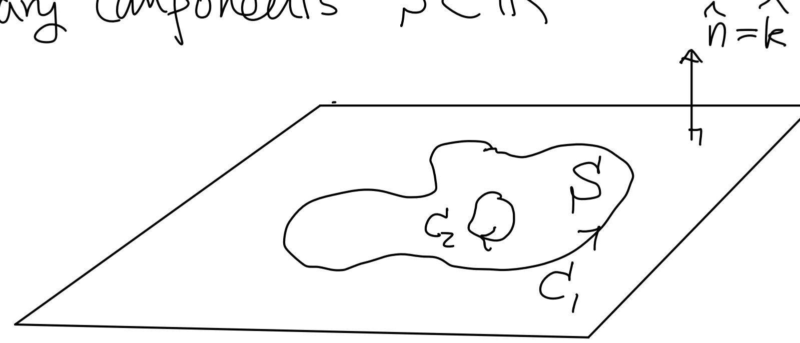
(3) (cont'd)

2 boundary components



(4) 2 boundary components $S \subset \mathbb{R}^2$



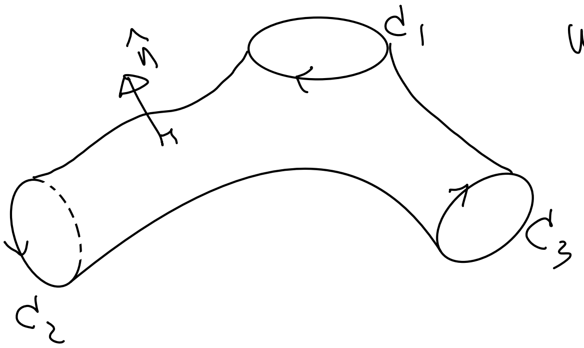
(Ex!)

<u>Important remark</u>: If S is a region in $\mathbb{R}^2$, then a boundary component of S (C_1 or C_2 for instance) has "2" concepts of "oriented anti-clockwisely" with respect to S = region and $\mathbb{R}^2$.

Even S and $\mathbb{R}^2$ have the same orientation, i.e. $\hat{n} = \hat{k}$, we still have the following situations: (C_1, C_2 as in figure above)

	S (region)	$\mathbb{R}^2$
C_1	anti-clockwise (+)	anti-clockwise (+)
C_2	anti-clockwise (+)	clockwise (−)

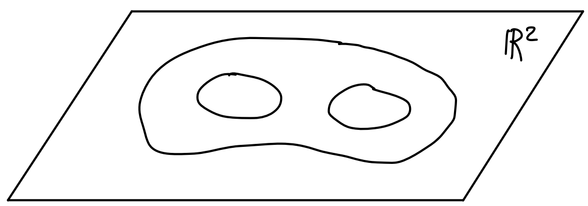
(5)



(Ex!)

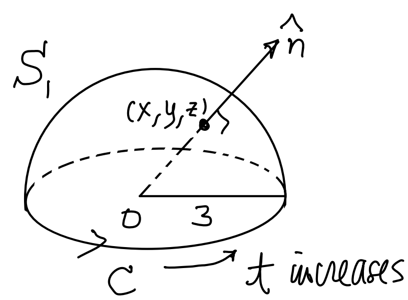
what is the oriented of C_i s.t. their oriented anti-clockwise with respect to $\hat{n}$? (Ex!)

(Similarly optional:



?)

eg 61 Verifying Stokes' Thm



(a) $S_1 : x^2+y^2+z^2=9,\ z\geq 0$

with upward unit normal $\hat{n}$ (i.e. $\hat{k}$-component >0) (pointing away from the origin)

Then boundary $C: x^2+y^2=9,\ z=0$ with anti-clockwise wrt $\hat{n}$.

Let $\vec{F}=y\hat{i}-x\hat{j}$, and we want to calculate

$$\iint_{S_1}(\vec{\nabla}\times\vec{F})\cdot\hat{n}\,d\sigma \quad\text{and}\quad \oint_C \vec{F}\cdot d\vec{r}$$

Parametrize C by $\vec{r}(t)=(3\cos t, 3\sin t, 0)$, $0\leq t\leq 2\pi$
$= 3\cos t\,\hat{i}+3\sin t\,\hat{j}$

has the correct direction, i.e. oriented anti-clockwisely wrt $\hat{n}$.

$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} (3\sin t\,\hat{i} - 3\cos t\,\hat{j})\cdot(-3\sin t\,\hat{i} + 3\cos t\,\hat{j})\,dt$$

$$= -18\pi \quad \text{(check!)}$$

For the surface integral

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = -2\hat{k} \qquad \text{(Check!)}$$

Since S_1 is a hemisphere (upper) centered at origin of radius 3,

$$\hat{n} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{3} \qquad \left(\begin{array}{c} z\geq 0 \\ \Downarrow \\ \text{upward} \end{array}\right)$$

The surface S_1 can be regarded as a level surface given by

$$g(x,y,z) = x^2+y^2+z^2 = 9$$

$$\Rightarrow \quad \vec{\nabla}g = (2x, 2y, 2z)$$

Since $z>0$ (except the bdy) on S_1, $\frac{\partial g}{\partial z} = 2z \neq 0$

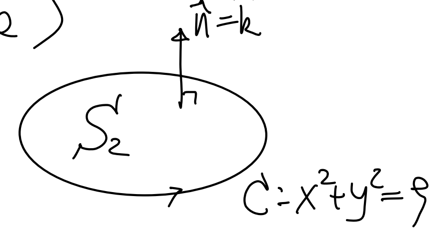
Hence $$d\sigma = \frac{|\vec{\nabla}g|}{\left|\frac{\partial g}{\partial z}\right|}\,dx\,dy = \frac{\sqrt{4x^2+4y^2+4z^2}}{|2z|}\,dx\,dy = \frac{3}{|z|}\,dx\,dy$$

Therefore $$\iint_{S_1} (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,d\sigma = \iint_{\{x^2+y^2\leq 9\}} (-2\hat{k})\cdot\frac{x\hat{i}+y\hat{j}+z\hat{k}}{3}\,\frac{3}{|z|}\,dx\,dy$$

$$= \iint_{\{x^2+y^2\leq 9\}} (-2)\,dx\,dy = -18\pi \qquad \#$$

(b) (Same C & same $\vec{F}$, but new surface)

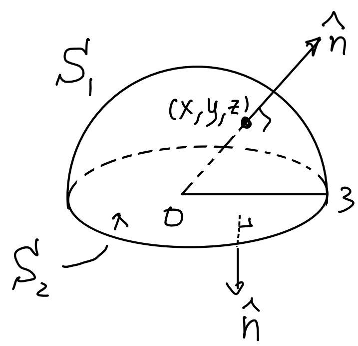
$S_2 : x^2+y^2<9, \quad z=0$ with $\hat{n}=\hat{k}$



$$\iint_{S_2} (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,d\sigma = \iint_{\{x^2+y^2<9\}} (-2\hat{k})\cdot\hat{k}\,dx\,dy = -18\pi$$

(c) Same $\vec{F} = y\hat{i} - x\hat{j}$

$S_3 = S_1 \cup S_2$



S_3 has no boundary and in fact encloses a solid region

Suppose $\hat{n}$ = outward unit normal of the solid

$$\iint_{S_3} (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,d\sigma = \iint_{S_1} (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,d\sigma + \iint_{S_2} (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,d\sigma$$

(same as eg (a)) (opposite to eg (b))

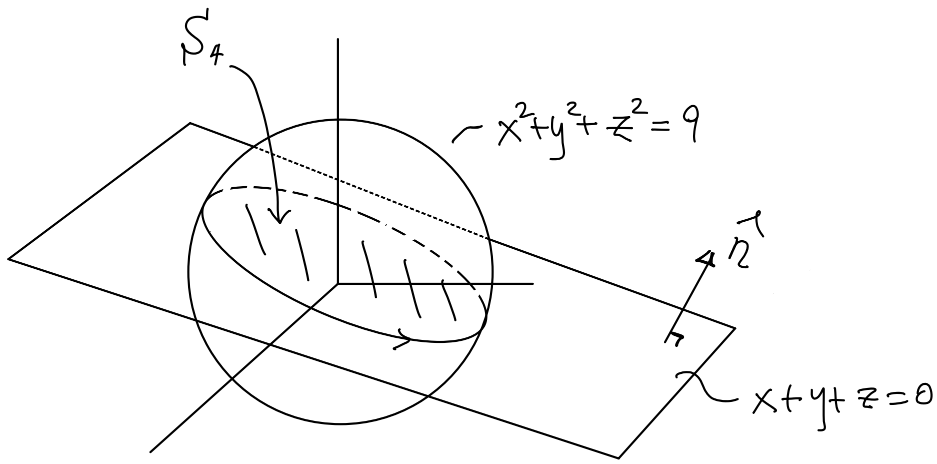
$$= \iint_{S} (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,d\sigma + \iint_{S_2} (\vec{\nabla}\times\vec{F})\cdot(-\hat{k})\,d\sigma$$

$$= -18\pi - (-18\pi) = 0$$

$$\left(= \oint_C \vec{F}\cdot d\vec{r} - \oint_C \vec{F}\cdot d\vec{r} = 0 \right)$$

\#

eg62 Let $\vec{F} = y\hat{i} - x\hat{j}$ (same $\vec{F}$ as in eg61, new surface & new boundary curve)



$S_4 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \leq 9,\ x+y+z=0\}$ with upward unit normal ($\hat{k}$-component ≥ 0)

boundary curve of S_4 : $C_4 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 9,\ x+y+z=0\}$

Find $\oint_{C_4} \vec{F} \cdot d\vec{r}$ (with direction of C_4 given as in the figure.)

Soln: $\hat{n}$ is normal to S_4 $\Rightarrow$ $\hat{n} = \dfrac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$ is upward

Apply Stokes' Thm

$$\oint_{C_4} \vec{F} \cdot d\vec{r} = \iint_{S_4} (\vec{\nabla} \times \vec{F}) \cdot \hat{n}\, d\sigma$$

$$= \iint_{S_4} (-2\hat{k}) \cdot \frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}\, d\sigma$$

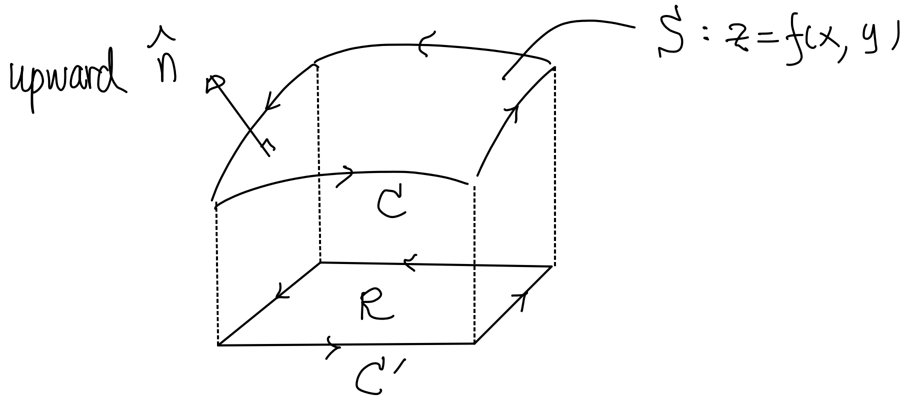
$$= -\frac{2}{\sqrt{3}} \iint_{S_4} d\sigma = -\frac{2}{\sqrt{3}} \text{Area}(S_4)$$

$$= -\frac{2}{\sqrt{3}} (\pi \cdot 3^2) = -\frac{18\pi}{\sqrt{3}} \quad \#$$

Proof of Stokes' Thm

Special case: S is a graph given by

$z = f(x,y)$ over a region R with upward normal



Assume C is the boundary of S, and C' is the boundary of R (anti-clockwisely oriented wrt the unit normal of S, and the plane (with normal $\hat{k}$) respectively)

Parametrize the graph as

$$\vec{r}(x,y) = x\hat{i} + y\hat{j} + f(x,y)\hat{k}, \qquad (x,y) \in R$$

Then as before $\begin{cases} \vec{r}_x = \hat{i} + f_x\hat{k} \\ \vec{r}_y = \hat{j} + f_y\hat{k} \end{cases}$

$$\Rightarrow \quad \vec{r}_x \times \vec{r}_y = -f_x\hat{i} - f_y\hat{j} + \hat{k}$$ +ve $\Rightarrow$ upward.

Hence $\hat{n} = \dfrac{\vec{r}_x \times \vec{r}_y}{\|\vec{r}_x \times \vec{r}_y\|}$ is the upward unit normal of S.

and $d\sigma = \|\vec{r}_x \times \vec{r}_y\|\, dxdy = \|\vec{r}_x \times \vec{r}_y\|\, dA$ area element of R.

Let $\vec{F} = M\hat{i} + N\hat{j} + L\hat{k}$ be the C^1 vector field.

Then $$\iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\, d\sigma = \iint_R (\vec{\nabla}\times\vec{F})(\vec{r}(x,y))\cdot \frac{\vec{r}_x\times\vec{r}_y}{\cancel{\|\vec{r}_x\times\vec{r}_y\|}} \cancel{\|\vec{r}_x\times\vec{r}_y\|}\, dA$$

$$= \iint_R \left[(L_y - N_z)\hat{i} + (M_z - L_x)\hat{j} + (N_x - M_y)\hat{k}\right]\cdot(-f_x\hat{i} - f_y\hat{j} + \hat{k})\, dA$$

$$= \iint_R \left[-f_x(L_y - N_z) - f_y(M_z - L_x) + (N_x - M_y)\right] dA$$

For the line integral

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C M\,dx + N\,dy + L\,dz$$

$$= \oint_{C'} M\,dx + N\,dy + L\,df \qquad (z = f(x,y))$$

$$= \oint_{C'} (M\,dx + N\,dy) + L(f_x\,dx + f_y\,dy)$$

$$= \oint_{C'} (M + f_x L)\,dx + (N + f_y L)\,dy$$

Precisely: If C' is parametrized by $\vec{\gamma}(t) = (x(t), y(t))$, $a \le t \le b$

Then C is parametrized by

$$\vec{r}(t) = (x(t), y(t), f(x(t), y(t))), \quad a \le t \le b$$

$$\Rightarrow \oint_C \vec{F}\cdot d\vec{r} = \int_a^b \Big[M(\vec{r}(t))\,x'(t) + N(\vec{r}(t))\,y'(t) + L(\vec{r}(t))\,\frac{d}{dt} f(x(t), y(t))\Big]\,dt$$

$$= \int_a^b \left[M x' + N y' + L(f_x x' + f_y y')\right] dt$$

$$= \int_a^b \left[(M + f_x L)x' + (N + f_y L)y'\right] dt$$

$$= \oint_{C'} (M + f_x L)\,dx + (N + f_y L)\,dy$$

Then by Green's Thm

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_{C'} (M + f_x L)\,dx + (N + f_y L)\,dy$$

$$= \iint_R \left[\frac{\partial}{\partial x}\left(N + f_y L\right) - \frac{\partial}{\partial y}\left(M + f_x L\right) \right] dA$$

$$= \iint_R \left[\begin{array}{l} \frac{\partial}{\partial x}\left(N(x,y,f(x,y)) + f_y(x,y)\, L(x,y,f(x,y))\right) \\ - \frac{\partial}{\partial y}\left(M(x,y,f(x,y)) + f_x(x,y)\, L(x,y,f(x,y))\right) \end{array} \right] dA$$

$$= \iint_R \left[\begin{array}{l} (N_x + N_z f_x) + \left(\cancel{f_{yx} L} + f_y (L_x + \cancel{L_z f_x})\right) \\ - (M_y + M_z f_y) - \left(\cancel{f_{xy} L} + f_x (L_y + \cancel{L_z f_y})\right) \end{array} \right] dA$$

(assume $f \in C^2$)

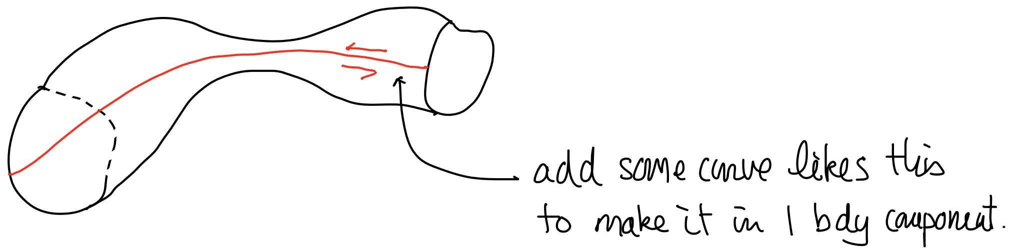
$$= \iint_R \left[-f_x (L_y - N_z) - f_y (M_z - L_x) + (N_x - M_y) \right] dA$$

$$= \iint_S (\vec{\nabla} \times \vec{F}) \cdot \hat{n}\, d\sigma$$

This proves the case for C^2 graph.

General case: Divides S into finitely many pieces which are graphs (in certain projection).

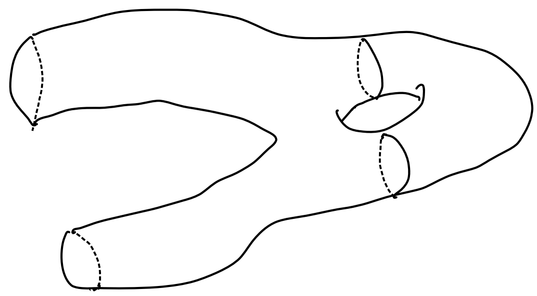
[This includes S with many boundary components as in the Green's Thm



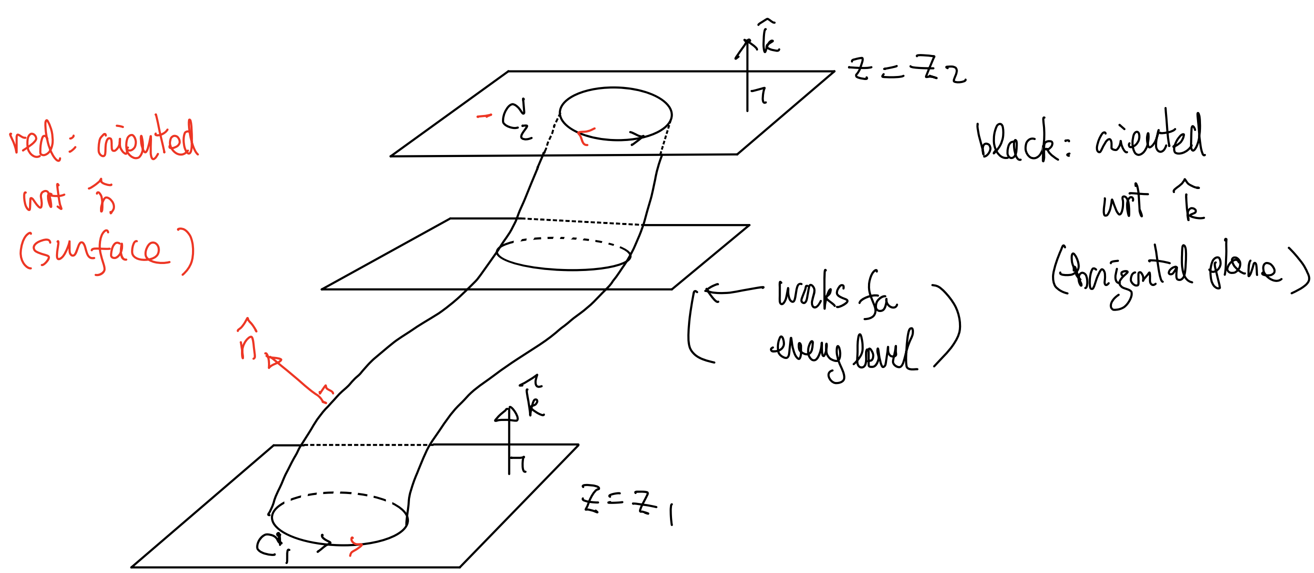
]

(Proof of general case omitted)

Note: Stokes' Thm applies to surfaces like the following



eg 63: Let $\vec{F}$ be a vector field such that $\vec{\nabla}\times\vec{F}=\vec{0}$ and defined on a region containing the surface S with unit normal vector field $\hat{n}$ as in the figure:



The boundary C of S has 2 components C_1 & C_2 at the level $z = z_1$ & $z = z_2$ respectively.

If both C_1, C_2 oriented anti-clockwisely with respect to the "horizontal planes" (i.e. $\hat{k}$)

Then when C oriented anti-clockwisely with respect to the surface normal $\hat{n}$, we have

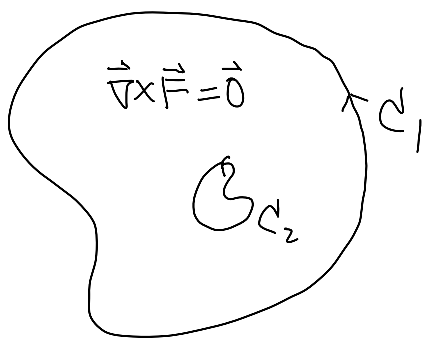
$$C = C_1 - C_2 \quad (C_1 \cup (-C_2))$$

And Stokes' Thm $\Rightarrow$

$$0 = \iint_S (\vec{\nabla} \times \vec{F}) \cdot \hat{n}\, d\sigma = \oint_C \vec{F} \cdot d\vec{r} = \oint_{C_1} \vec{F} \cdot d\vec{r} - \oint_{C_2} \vec{F} \cdot d\vec{r}$$

$$\Rightarrow \quad \oint_{C_1} \vec{F} \cdot d\vec{r} = \oint_{C_2} \vec{F} \cdot d\vec{r} \qquad ※$$

Compare this with Green's Thm on plane region with one hole :



$$\Rightarrow \quad \oint_{C_1} \vec{F} \cdot d\vec{r} = \oint_{C_2} \vec{F} \cdot d\vec{r} \qquad \text{(check!)}$$

anti-clockwisely wrt "plane" (not see region as an oriented surface).