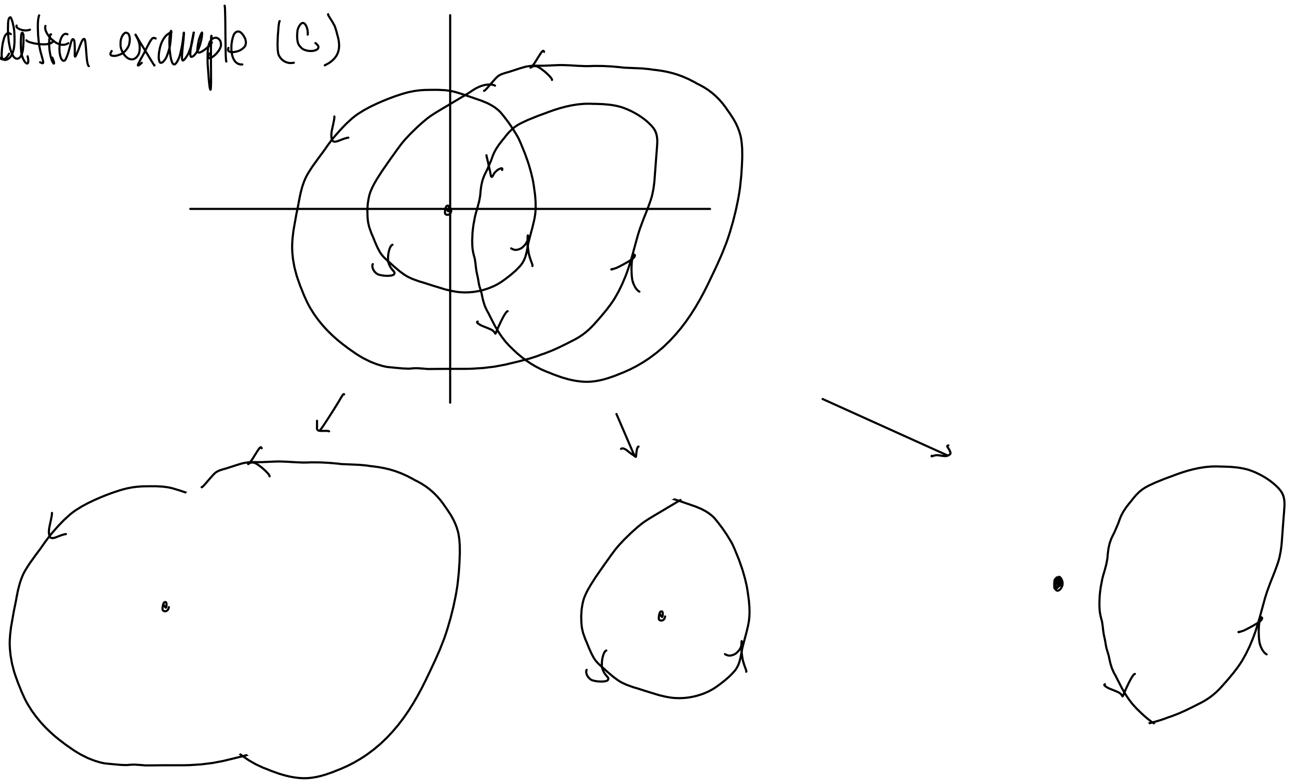
Addition example (c)



$\oint \cdots = 2\pi$ $\qquad$ $\oint \cdots = 2\pi$ $\qquad$ $\oint \cdots = 0$

Hence $\oint_C \vec{F} \cdot d\vec{r} = 2\pi + 2\pi + 0 = 4\pi$

(optional ex! : think of some examples with -2π)

Surface Area & Integral

$\vec{r}(t) \in \mathbb{R}^3$, $t \in [a,b]$

Def 14 Parametric Surface (Surface with parametrization)

A parametric surface (or a parametrization of a surface) in $\mathbb{R}^3$ is a continuous mapping of 2-variables into $\mathbb{R}^3$:

$$\vec{r}(u,v) = x(u,v)\,\hat{i} + y(u,v)\,\hat{j} + z(u,v)\,\hat{k}$$

And it is called smooth if

(1) $\vec{r}$ is C^1 (i.e. $x_u, x_v, y_u, y_v, z_u, z_v$ are continuous)

(2) $\boxed{\vec{r}_u \times \vec{r}_v \neq \vec{0}}$ $\forall\, u, v$

where

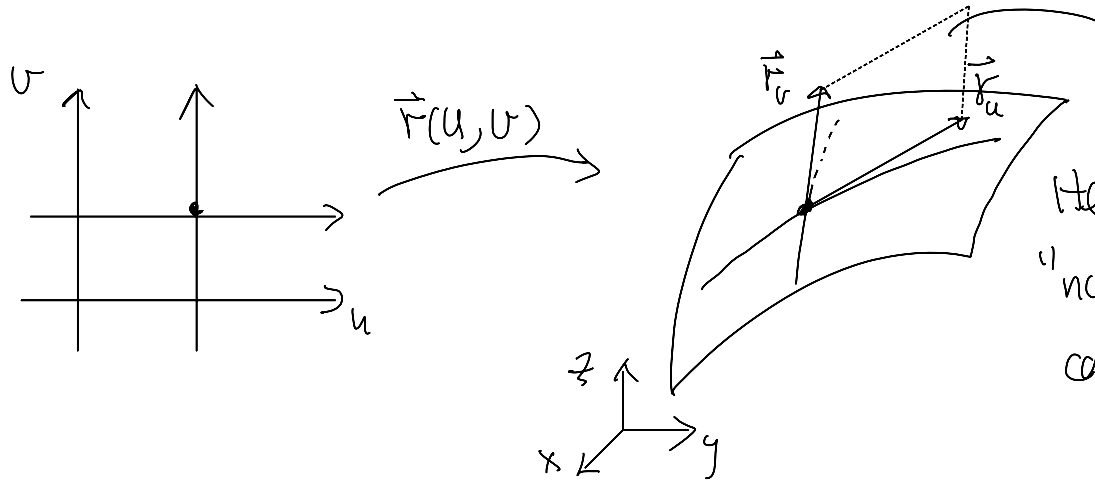
$$\begin{cases} \vec{r}_u = \dfrac{\partial \vec{r}}{\partial u} = \dfrac{\partial x}{\partial u}\hat{i} + \dfrac{\partial y}{\partial u}\hat{j} + \dfrac{\partial z}{\partial u}\hat{k} \\[2ex] \vec{r}_v = \dfrac{\partial \vec{r}}{\partial v} = \dfrac{\partial x}{\partial v}\hat{i} + \dfrac{\partial y}{\partial v}\hat{j} + \dfrac{\partial z}{\partial v}\hat{k} \end{cases}$$

$$\left(\begin{cases} \vec{r}_u = x_u\hat{i} + y_u\hat{j} + z_u\hat{k} \\ \vec{r}_v = x_v\hat{i} + y_v\hat{j} + z_v\hat{k} \end{cases} \right)$$

Note: Condition (2) $\Rightarrow$ $\vec{r}_u, \vec{r}_v$ are linearly independent

$\Rightarrow$ $\mathrm{span}\{\vec{r}_u, \vec{r}_v\}$ is in fact a 2-dim'l subspace.

$\Rightarrow$ "surface" cannot be degenerated to a curve or a point.

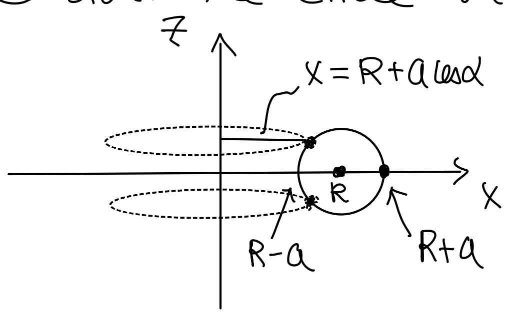


$= |\vec{r}_u \times \vec{r}_v| \neq 0$

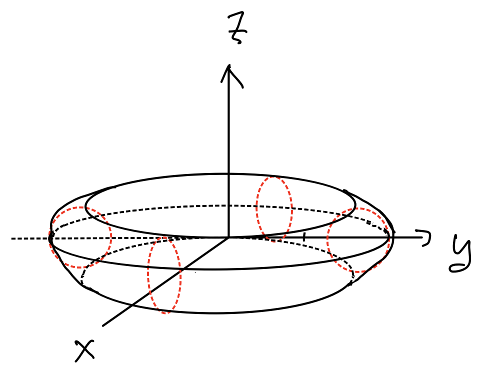
Hence, it is a "non-degenerate" condition.

eg51 (Torus)

Consider the circle on the xz-plane (ie. $y=0$) with radius $a>0$ centered at $(x,z)=(R,0)$ with $R>a$.



A parametrization is
$$\begin{cases} x = R + a\cos\alpha \\ z = a\sin\alpha \end{cases} \qquad \alpha\in[0,2\pi]$$



rotating the circle, we have a Torus

Then the parametrization of the Torus is
$$\begin{cases} x = (R+a\cos\alpha)\cos\theta \\ y = (R+a\cos\alpha)\sin\theta \\ z = a\sin\alpha \end{cases}, \qquad \begin{matrix}\alpha\in[0,2\pi] \\ \theta\in[0,2\pi]\end{matrix}$$

Ex: Check that it is a smooth surface:

It is clearly C^1, need to check

$$(x_\alpha, y_\alpha, z_\alpha) \times (x_\theta, y_\theta, z_\theta) \neq \vec{0}$$

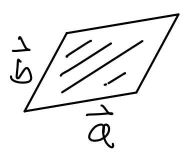
(See next example)

Note: This torus can also be described as

$$\left(\sqrt{x^2+y^2} - R\right)^2 + z^2 = a^2 \qquad (\text{Ex!})$$

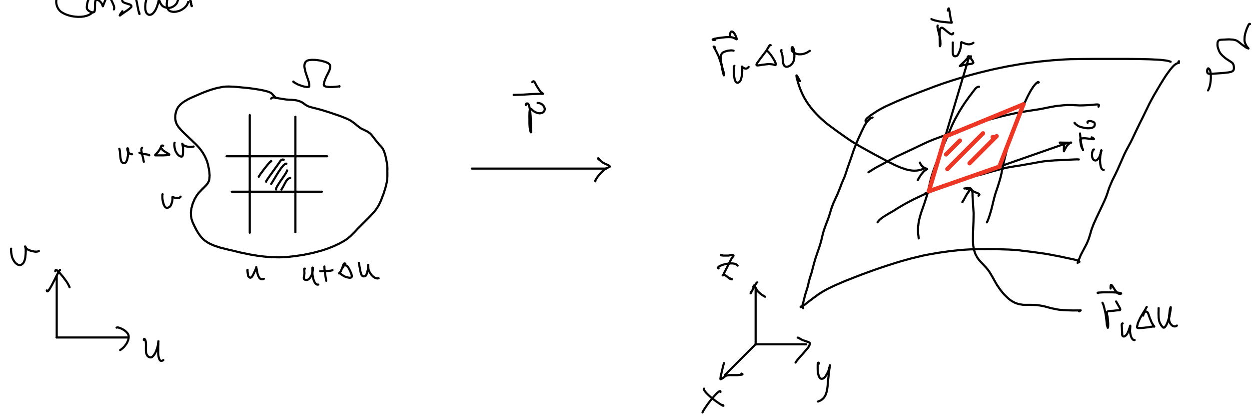
Surface Area

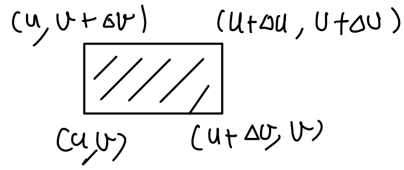
Recall: for $\vec{a}, \vec{b} \in \mathbb{R}^3$

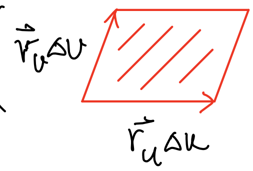
$|\vec{a} \times \vec{b}| = \|\vec{a} \times \vec{b}\| = \text{Area}($  $)$

Let $\vec{r}(u,v)$ be a parametrization of a surface S with $(u,v) \in \Omega$

Consider



$\Rightarrow$ "Area" on the surface corresponding to 

is approx. = Area()

$$= \left| (\vec{r}_u \Delta u) \times (\vec{r}_v \Delta v) \right|$$

$$= |\vec{r}_u \times \vec{r}_v| \, \Delta u \Delta v$$

Hence "Area element" of S, denoted by $d\sigma$, is given by

$$d\sigma = |\vec{r}_u \times \vec{r}_v| \, du\, dv$$

$$\boxed{d\sigma = |\vec{r}_u \times \vec{r}_v| \, dA}$$

dA: area element in the (u,v)-space.

Therefore, we make the following

<u>Def 15</u>: Let $S \subset \mathbb{R}^3$ be a <u>smooth</u> <u>parametric</u> surface given by $\vec{r}(u,v)$ for $(u,v) \in \Omega \subset \mathbb{R}^2$. Then

$$\text{Area}(S) \overset{\text{def}}{=\!=} \iint_\Omega |\vec{r}_u \times \vec{r}_v| \, dA$$

$$= \iint_\Omega \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| dA$$

(i.e. $\text{Area}(S) = \iint_\Omega d\sigma$)

eg 52: Surface area of torus given by ($R>a>0$ are constants)

$$\begin{cases} x = (R+a\cos\alpha)\cos\theta \\ y = (R+a\cos\alpha)\sin\theta \\ z = a\sin\alpha \end{cases}, \quad \begin{array}{l} \alpha \in [0,2\pi] \\ \theta \in [0,2\pi] \end{array}$$

Soln:

$$\vec{r}(\alpha,\theta) = (R+a\cos\alpha)\cos\theta\,\hat{i} + (R+a\cos\alpha)\sin\theta\,\hat{j} + a\sin\alpha\,\hat{k}$$

$$\begin{cases} \vec{r}_\alpha = -a\sin\alpha\cos\theta\,\hat{i} - a\sin\alpha\sin\theta\,\hat{j} + a\cos\alpha\,\hat{k} \\ \\ \vec{r}_\theta = -(R+a\cos\alpha)\sin\theta\,\hat{i} + (R+a\cos\alpha)\cos\theta\,\hat{j} \end{cases}$$

&

$$\begin{aligned} \vec{r}_\alpha \times \vec{r}_\theta = & -a(R+a\cos\alpha)\cos\theta\cos\alpha\,\hat{i} \\ & -a(R+a\cos\alpha)\sin\theta\cos\alpha\,\hat{j} \qquad \text{(Check!)} \\ & -a(R+a\cos\alpha)\sin\alpha\,\hat{k} \end{aligned}$$

$$\Rightarrow \quad |\vec{r}_\alpha \times \vec{r}_\theta| = a(R+a\cos\alpha) > 0 \qquad \text{(Check!)}$$

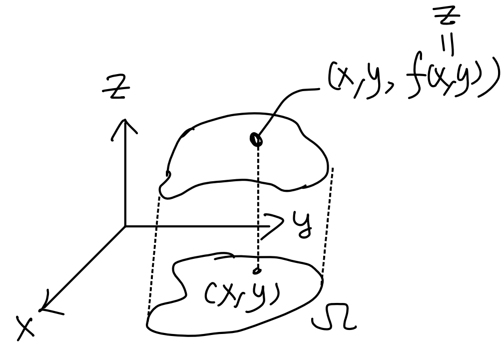
(So, the surface is "smooth")

$$\begin{aligned} \text{Area(Torus)} &= \iint_\Omega |\vec{r}_\alpha \times \vec{r}_\theta|\,dA \\ &= \int_0^{2\pi}\int_0^{2\pi} a(R+a\cos\alpha)\,d\alpha\,d\theta \\ &= 4\pi^2 Ra \qquad \text{(Check!)} \end{aligned}$$

#

Surface area of a graph

$z = f(x,y)$, $(x,y) \in \Omega$



Choose the following "natural" parametrization of the graph

$$\vec{r}(x,y) = x\hat{i} + y\hat{j} + f(x,y)\hat{k}$$

$$\Rightarrow \quad \begin{cases} \vec{r}_x = \hat{i} + f_x \hat{k} \\ \vec{r}_y = \hat{j} + f_y \hat{k} \end{cases}$$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = -f_x \hat{i} - f_y \hat{j} + \hat{k}$$

$$\Rightarrow \quad |\vec{r}_x \times \vec{r}_y| = \sqrt{f_x^2 + f_y^2 + 1} = \sqrt{1 + |\vec{\nabla} f|^2} \geq 1$$

(never zero $\Rightarrow$ "smooth" provided $f \in C^1$)

Thm 11: The surface area of a C^1 graph S given by

$$z = f(x,y), \quad (x,y) \in \Omega \subset \mathbb{R}^2$$

is

$$\text{Area}(S) = \iint_\Omega \sqrt{1 + |\vec{\nabla} f|^2}\, dA = \iint_\Omega \sqrt{1 + f_x^2 + f_y^2}\, dA$$

(Similarly for $x = f(y,z)$ or $y = f(x,z)$)

Implicit Surface (Level surface)

Suppose S is given by $F(x,y,z) = c$

i.e. $S = F^{-1}(c)$

(Note: F is a function of 3-variables, not vector field)

eg 53: $F(x,y,z) = x^2+y^2+z^2$

Is $F^{-1}(0)$ a surface?

Soln: No, since $F^{-1}(0) = \{(0,0,0)\}$ is a point, not "surface".

Remark: If $\vec{\nabla} F \neq \vec{0}$ at a point, then IFT implies that

$S = F^{-1}(c)$ is a "surface" (c = value of F at that point)

near that point (in fact, a graph!)

eg 53 (cont'd) $\vec{\nabla} F = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

$\vec{\nabla} F = \vec{0} \iff (x,y,z) = (0,0,0)$

Hence if $c > 0$, then $\forall (x,y,z) \in F^{-1}(c)$, we have

$\vec{\nabla} F(x,y,z) \neq \vec{0}$. $\quad (F^{-1}(\text{negative}) = \emptyset)$

$\Rightarrow S = F^{-1}(c)$ is a surface $\forall c > 0$.

(What are these surfaces?)

Terminology: $S = F^{-1}(c)$ is said to be <u>smooth</u> if (1) F is C^1 on S, and (2) $\vec{\nabla} F \neq \vec{0}$ on S.

How to compute surface area for a smooth level surface $S = F^{-1}(c)$?

By $\vec{\nabla} F \neq \vec{0}$, at least one of the partial derivative F_x, F_y or F_z is nonzero.

Let assume $F_z = \frac{\partial F}{\partial z} \neq 0$ (the other cases are similar)

IFT $\Rightarrow$ $S = F^{-1}(c) = \{F(x,y,z) = c\}$ can be written (locally) as a graph

$z = f(x,y)$ (near a pt.)

i.e. $F(x, y, f(x,y)) = c$,

$\forall (x,y)$ in some domain $\Omega \subset \mathbb{R}^2$ s.t. $(x,y,f(x,y))$ near "the pt"

Then Chain rule $\Rightarrow$ $\begin{cases} f_x = -\dfrac{F_x}{F_z} \\ f_y = -\dfrac{F_y}{F_z} \end{cases}$ $(F_z \neq 0)$

Hence $\text{Area}(S) = \iint_\Omega \sqrt{1 + |\vec{\nabla} f|^2}\, dA = \iint_\Omega \sqrt{1 + \frac{F_x^2}{F_z^2} + \frac{F_y^2}{F_z^2}}\, dA$

$$= \iint_\Omega \frac{\sqrt{F_x^2 + F_y^2 + F_z^2}}{|F_z|}\, dA$$

Thm 12 If $S = F^{-1}(c)$ is a <u>smooth</u> level surface such that $F_z \neq 0$, and can be represented by an implicit function over a domain Ω.

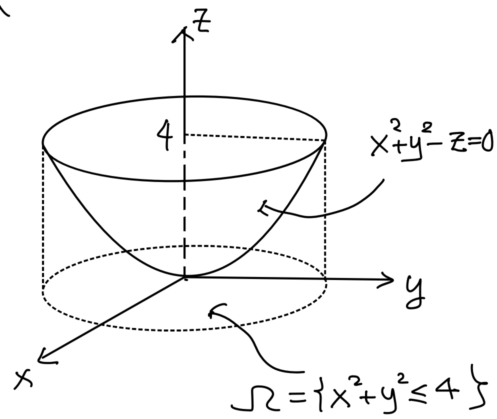
Then
$$\text{Area}(S) = \iint_{\Omega} \frac{|\vec{\nabla} F|}{|F_z|}\, dA = \iint_{\Omega} \frac{|\vec{\nabla} F|}{|F_z|}\, dx\,dy$$

(Similar for the cases that $F_x \neq 0$ or $F_y \neq 0$)

eg 54: Find surface area of the paraboloid

$x^2+y^2-z=0$ below $z=4$

(This is infact a graph, but we do it using method of level surface)



Soln: Let $F(x,y,z) = x^2+y^2-z$

For $z=4$, $x^2+y^2=4$

$\Rightarrow$ projected region $\Omega = \{(x,y) : x^2+y^2 \le 4\}$

$\vec{\nabla} F = 2x\,\hat{i} + 2y\,\hat{j} - \hat{k}$

$\therefore \quad F_z = -1 \neq 0, \quad \forall (x,y)\in\Omega$

$$\Rightarrow \text{Surface Area} = \iint_{\Omega} \frac{|\vec{\nabla} F|}{|F_z|}\, dA = \iint_{\Omega} \frac{\sqrt{4x^2+4y^2+1}}{|-1|}\, dx\,dy$$

$$= \iint_{\{x^2+y^2\le 4\}} \sqrt{4x^2+4y^2+1}\, dx\,dy = \frac{\pi}{6}\left[(\sqrt{17})^3 - 1\right]$$

#

Def 16 Surface Integral (of a function)

Suppose $G : S \to \mathbb{R}$ is a continuous function on a surface S, parametrized by $\vec{r}(u,v)$, $(u,v) \in R$ (region R). Then the integral of G on S is

$$\iint_S G\, d\sigma \overset{\text{def}}{=\!=} \iint_R G(\vec{r}(u,v))\, |\vec{r}_u \times \vec{r}_v|\, dA$$

area element of S

element area of the parameter space $dA = du\,dv$

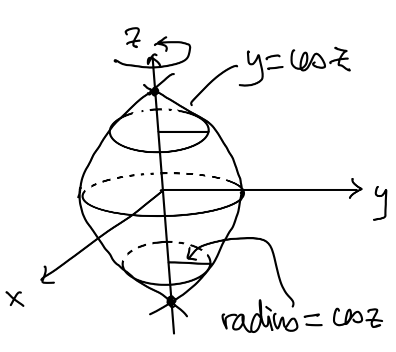
Note: In the cases of graph or level surface, we have

(i) $$\iint_S G\, d\sigma = \iint_{(x,y)} G(x,y,f(x,y))\sqrt{1+|\vec{\nabla} f|^2}\, dx\,dy$$ (for $z = f(x,y)$)

(ii) $$\iint_S G\, d\sigma = \iint_{(x,y)} G(x,y,z)\, \frac{|\vec{\nabla} F|}{|F_z|}\, dx\,dy$$ (for $F(x,y,z) = c$, $F_z \neq 0$)

(may be difficult to find these: region & z in terms of (x,y))

eg 56 (a surface of revolution of the curve $y = \cos z$)



$\left(-\frac{\pi}{2} \le z \le \frac{\pi}{2}\right)$

Let $G(x,y,z) = \sqrt{1-x^2-y^2}$ be a function on S

Find $\iint_S G\, d\sigma$.

Soln: S can be parametrized by

$$\begin{cases} x = \cos z \cos\theta \\ y = \cos z \sin\theta \\ z = z \end{cases} \qquad \begin{array}{l} \theta \in [-\pi, \pi] \\ z \in [-\frac{\pi}{2}, \frac{\pi}{2}] \end{array}$$

(Solve from the $y = \cos z = 0$)

i.e. $\vec{r}(\theta, z) = \cos z \cos\theta\,\hat{i} + \cos z \sin\theta\,\hat{j} + z\hat{k}$

(Note that there is an exceptional set of "1-dim" so that $\vec{r}$ is not one-to-one, or not smooth corresponds to $\theta = \pi$ or $-\pi$, $z = -\frac{\pi}{2}$ or $\frac{\pi}{2}$)

$$\begin{cases} \vec{r}_\theta = -\cos z \sin\theta\,\hat{i} + \cos z \cos\theta\,\hat{j} \\ \vec{r}_z = -\sin z \cos\theta\,\hat{i} - \sin z \sin\theta\,\hat{j} + \hat{k} \end{cases}$$

(check!)

$\vec{r}_\theta \times \vec{r}_z = \cos z \cos\theta\,\hat{i} + \cos z \sin\theta\,\hat{j} + \sin z \cos z\,\hat{k}$ (check!)

$\Rightarrow |\vec{r}_\theta \times \vec{r}_z| = \sqrt{\cos^2 z\,(1 + \sin^2 z)} = \cos z \sqrt{1 + \sin^2 z}$ (check!)

($\cos z \geq 0$ since $z \in [-\frac{\pi}{2}, \frac{\pi}{2}]$)

Then $\displaystyle\iint_S G\,d\sigma = \int_{-\pi}^{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} G(\vec{r}(\theta, z))\,|\vec{r}_\theta \times \vec{r}_z|\,dz\,d\theta$

$$= \int_{-\pi}^{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1 - \cos^2 z}\,\cos z \sqrt{1 + \sin^2 z}\,dz\,d\theta$$

$$= 4\pi \int_0^{\frac{\pi}{2}} \sin z \cos z \sqrt{1 + \sin^2 z}\,dz$$

$$= \cdots = \frac{4\pi}{3}\left(2\sqrt{2} - 1\right)$$

(check!)

#