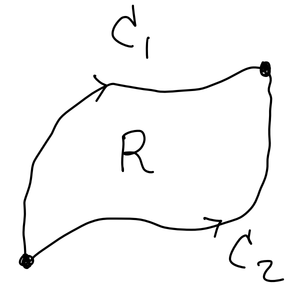
Pf of Thm 10 ($n=2$)

We only remain to show that if Ω is simply-connected (& connected) and $\vec{\nabla}\times\vec{F}=\vec{0}$ (i.e. $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, $\vec{F}=M\hat{i}+N\hat{j}$)

then $\vec{F}$ is conservative.



Case 1 C_1, C_2 have no intersection (except at the same end points)

Then "Ω is simply-connected"

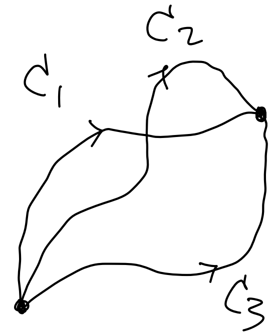
$\Rightarrow$ The region R enclosed by C_1 & C_2 lies completely inside Ω.

Then Green's Thm $\Rightarrow$

$$0=\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dA = \pm\oint_{C_1-C_2}(M\,dx+N\,dy)$$

$$\Rightarrow \int_{C_1} M\,dx+N\,dy = \int_{C_2} M\,dx+N\,dy$$

Case 2 C_1 & C_2 intersect



Pick another curve C_3 with the same starting point and end point, and doesn't intersect C_1 & C_2.

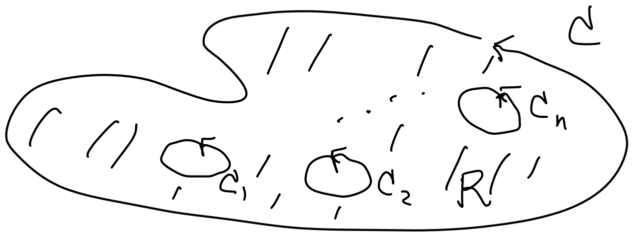
Then Case 1 $\Rightarrow$ $\int_{C_1} M\,dx+N\,dy = \int_{C_3} M\,dx+N\,dy = \int_{C_2} M\,dx+N\,dy$

$\therefore \int_C \vec{F}\cdot d\vec{r}$ is indep. of path & hence $\vec{F}$ is conservative. #

In order to apply Green's Thm to more general situations, we need

Thm (Green's Thm (general form))

Suppose that we have a simple closed curve C in $\mathbb{R}^2$



Suppose that $C_1, C_2, \cdots, C_n$ be pairwise disjoint, piecewise smooth, simple closed curves, such that $C_1, \cdots, C_n$ are enclosed by C.

(All $C, C_1, \cdots, C_n$ are anti-clockwise oriented.)

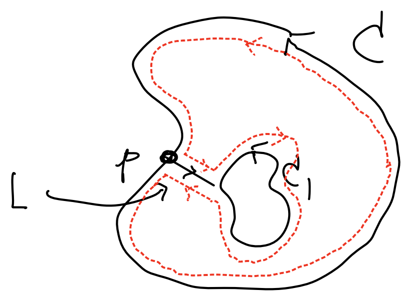
Let R be the region between C and $C_1, \cdots, C_n$.

Suppose that $\vec{F} = M\hat{i} + N\hat{j}$ is defined on some open set containing R, and is C^1. Then

$$\boxed{\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \oint_C M\,dx + N\,dy - \sum_{i=1}^{n} \oint_{C_i} M\,dx + N\,dy}$$

(This is the tangential form. The normal form is similar)

Sketch of Proof



For simplicity, only one C_1 inside C

We connect C & C_1 by an "arc" L

and consider the "simple" closed curve (starting from the pt p)

$$C^* = C + L - C_1 - L$$

Then the region R enclosed by C & C_1 is the region enclosed by C^* except the arc L.

Hence $$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \iint_{R\setminus L} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

$$\text{(Green's)} = \oint_{C^*} (M\,dx + N\,dy)$$

$$= \left(\oint_C + \int_L - \oint_{C_1} - \int_L\right)(M\,dx+N\,dy)$$

$$= \oint_C M\,dx+N\,dy - \oint_{C_1} M\,dx+N\,dy$$

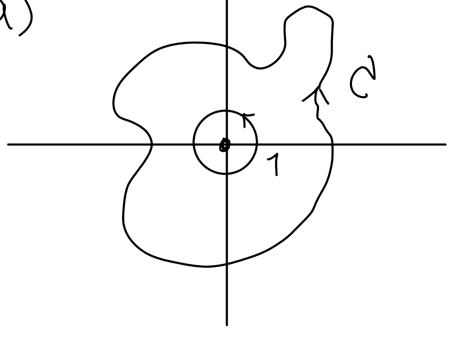
#

eg 49: $\vec{F} = \dfrac{-y}{x^2+y^2}\hat{i} + \dfrac{x}{x^2+y^2}\hat{j}$ on $\mathbb{R}^2\setminus\{(0,0)\} = \Omega$ ($=\Omega_2$ in eg 43)

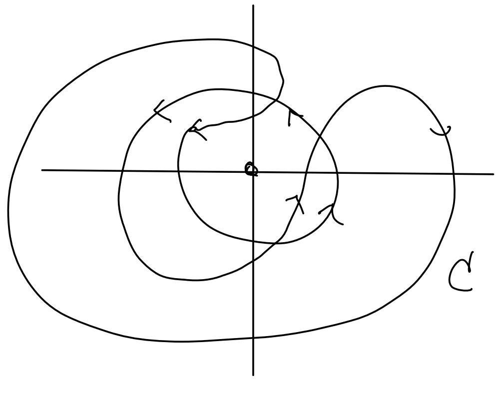
we've calculated $\oint_{C_1} \vec{F}\cdot d\vec{r} = 2\pi$ for $C_1: x^2+y^2=1$ (anti-clockwise)

How about

(a)



(b)



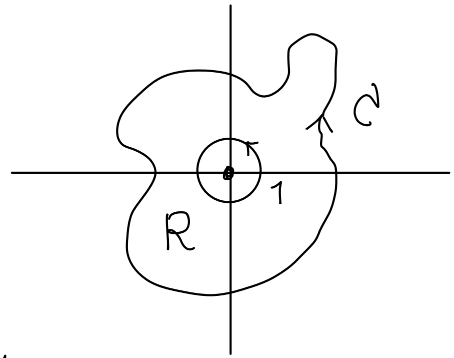
$\oint_C \vec{F}\cdot d\vec{r} = ?$

$\underline{\text{Soln}}$ (a) Recall that $\vec{\nabla} \times \vec{F} = \vec{0}$

(Note that Green's Thm doesn't apply directly, since C' encloses the origin $(0,0)$ where $\vec{F}$ is not defined)

Consider the region between

C' & C_1 (= unit circle)



By Green's Thm (general form)

$$\left(\oint_{C'} - \oint_{C_1} \right) \vec{F} \cdot d\vec{r} = \iint_R (\vec{\nabla} \times \vec{F}) \cdot \hat{k} \, dA$$

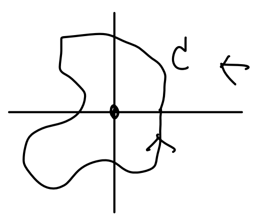
$$= 0$$

$$\Rightarrow \quad \oint_{C'} \vec{F} \cdot d\vec{r} = \oint_{C_1} \vec{F} \cdot d\vec{r} = 2\pi$$

In particular, $C_R = \{ x^2 + y^2 = R^2 \}$, $\forall R > 1$, we have $\oint_{C_R} \vec{F} \cdot d\vec{r} = 2\pi$

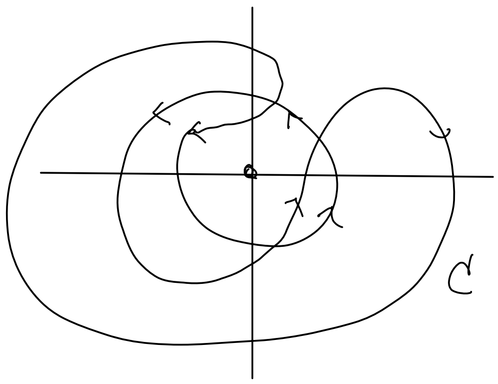
(Ex!) Same argument $\Rightarrow$ $\oint_{C_R} \vec{F} \cdot d\vec{r} = 2\pi$ for $0 < R < 1$

$\Rightarrow$ $\oint_{C'} \vec{F} \cdot d\vec{r} = 2\pi$ for any simply-closed curve enclosing the origin $(0,0)$

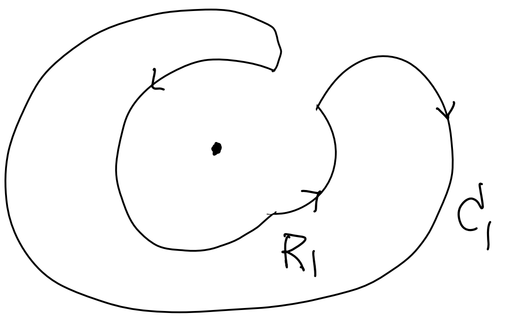


$\leftarrow$ can be small

(b)

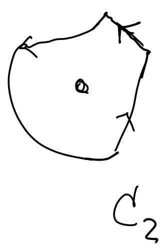


Decompose the curve into



doesn't enclose (0,0)

$$\oint_{C_1} \vec{F}\cdot d\vec{r} = \iint_{R_1} (\vec{\nabla}\times\vec{F})\cdot\vec{k}\, dA = 0$$



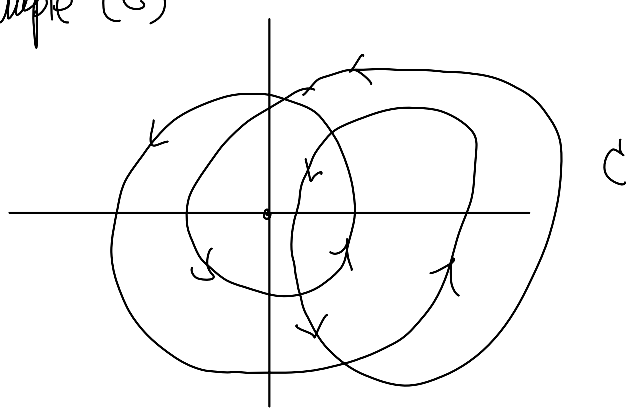
encloses (0,0)

(by (a)) $\oint_{C_2} \vec{F}\cdot d\vec{r} = 2\pi$

$$\therefore \quad \oint_{C} \vec{F}\cdot d\vec{r} = 0 + 2\pi = 2\pi$$

#

Addition example (c)



What is $\oint_{C} \vec{F}\cdot d\vec{r}$? (To be cont'd)