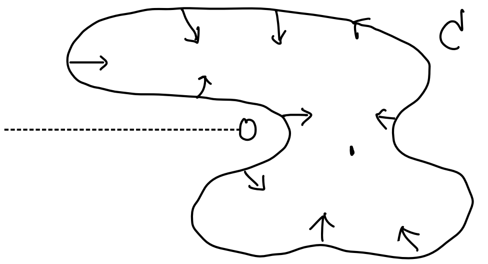
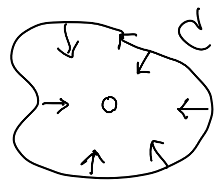
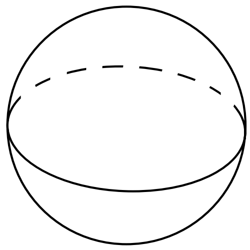
Summary

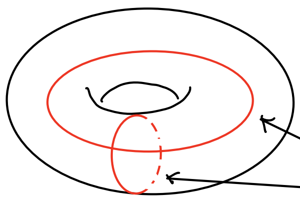
Ω_1	Ω_2
$f(x,y)=\theta$ "smooth" function on Ω_1	$f(x,y)=\theta$ is not a "smooth" function on Ω_2 (θ cannot be extended continuously on the whole Ω_2)
$C: x^2+y^2=1$ is not a curve in Ω_1 ($(-1,0)\in C$, but $(-1,0)\notin\Omega_1$)	$C: x^2+y^2=1$ is a closed curve in Ω_2
 Closed curves cannot circle around the origin $\Rightarrow$ closed curves can be "deformed" continuously (within Ω_1) to a point (in Ω_1)	 C encloses the "hole" $\Rightarrow$ C cannot be "deformed" continuously (within Ω_2) to a point (in Ω_2)

Def 15 A subset $\Omega\subset\mathbb{R}^n$, $n=2$ or 3, is called simply-connected if every closed curve in Ω can be contracted to a point in Ω without ever leaving Ω.

(contracted = deformed continuously)

eg44 Ω_1 in eg43 is simply-connected, but Ω_2 is not simply-connected,

eg45:  $S^2 \subset \mathbb{R}^3$ $S^2 = \{x^2+y^2+z^2=1\}$ (unit sphere) is simply-connected.

eg46:  torus $\mathbb{T}^2 \cong S^1 \times S^1 \subset \mathbb{R}^3$ is not simply-connected

those 2 closed curves cannot be contracted to a point on $\mathbb{T}^2$.

Remark: Simply-connectedness is a global condition to guarantee " "PDEs in Cor to Thm 9" $\Rightarrow$ "conservative" "

Thm 10: Suppose $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3, is connected and simply-connected. Let $\vec{F}$ be C^1 vector field on Ω. Then

$\vec{F}$ is conservative on Ω $\Leftrightarrow$ components of $\vec{F}$ satisfy the system of PDEs in the Cor to Thm 9.

(Pf: later)

eg 47 : Let $\Omega \equiv \mathbb{R}^3$ (connected and simply-connected)

Let $\vec{F} = M\hat{i} + N\hat{j} + L\hat{k}$

$$= (y+e^z)\hat{i} + (x+1)\hat{j} + (1+xe^z)\hat{k}.$$

Find the potential function f of $\vec{F}$, i.e. $\vec{\nabla} f = \vec{F}$.

Soln: That is, we want to solve

$$\frac{\partial f}{\partial x} = M, \quad \frac{\partial f}{\partial y} = N, \quad \frac{\partial f}{\partial z} = L$$

Checking M, N, L satisfy the system of PDEs in the Cor to Thm 9

$$\frac{\partial M}{\partial y} = 1 \qquad \frac{\partial M}{\partial z} = e^z$$

$$\frac{\partial N}{\partial x} = 1 \qquad \frac{\partial N}{\partial z} = 0$$

$$\frac{\partial L}{\partial x} = e^z \qquad \frac{\partial L}{\partial y} = 0$$

(No need to check $\frac{\partial M}{\partial x}, \frac{\partial N}{\partial y}, \frac{\partial L}{\partial z}$)

Thm 10 $\Rightarrow$ existence of potential function f.

To find f explicitly

$$\frac{\partial f}{\partial x} = M = y + e^z$$

$$f = \int (y+e^z)\,dx = x(y+e^z) + \text{"const. in } x\text{"}$$

$$= x(y+e^z) + g(y,z) \quad \text{for some function } g(y,z)$$

Then take $\frac{\partial}{\partial y}$: $\quad N = x+1 = \frac{\partial f}{\partial y} = x + \frac{\partial g}{\partial y}$

$$\Rightarrow \quad \frac{\partial g}{\partial y} = 1$$

$$\Rightarrow \quad g = \int 1\, dy = y + \text{"const. in } y\text{"}$$

$$= y + h(z) \quad \text{for some function of } z$$

$$\therefore \quad f = x(y+e^z) + y + h(z)$$

Then take $\frac{\partial}{\partial z}$: $\quad L = 1 + xe^z = \frac{\partial f}{\partial z} = xe^z + h'(z)$

$$\Rightarrow \quad h'(z) = 1 \quad \Rightarrow \quad h(z) = z + \text{const.}$$

Hence $\quad f(x,y,z) = x(y+e^z) + y + z + c$, where c = constant

are potential functions of $\vec{F}$. #

[In practice: $\quad (y+e^z)\hat{i} + (x+1)\hat{j} + (1+xe^z)\hat{k}$

$$\Leftrightarrow (y+e^z)dx + (x+1)dy + (1+xe^z)dz$$

$$= (y\,dx + x\,dy) + (e^z dx + xe^z dz) + dy + dz$$

$$= d(xy) + d(xe^z) + d(y+z)$$

$$= d(xy + xe^z + y + z) \quad \#$$
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