# Change of Variable Formula

## Review of 1-variable

In Riemann integral (sum)

$$\int_a^b f(x)\,dx = \int_{[a,b]} f(x)\,\underbrace{dx}_{\left(\sim |\Delta x| = \text{length of subinterval} > 0\right)}$$

$(a<b)$

$\uparrow$ as set (don't care about the direction)

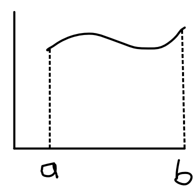
If $a>b$

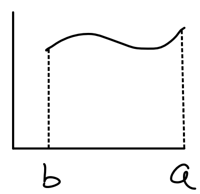
$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx = -\int_{[b,a]} f(x)\,dx$$

([b, a] & [a, b] represent the same set)

In summary

$$\int_a^b f(x)\,dx = \begin{cases} \displaystyle\int_{[a,b]} f(x)\,dx, & \text{if } a\le b \\ \displaystyle -\int_{[a,b]} f(x)\,dx, & \text{if } a\ge b \end{cases}$$




$\left( [a,b] = [b,a] \text{ as set} = \{x : x \text{ between } a \,\&\, b\} \right)$

## Change of variable in 1-variable

$$\int_a^b f(x)\,dx = \int_c^d \left[ f(x(u))\,\frac{dx}{du} \right] du$$

where $c = u(a)$, $d = u(b)$.

If $\frac{dx}{du} > 0$, then $d = u(b) > u(a) = c$

$$\therefore \int_a^b f(x)dx = \int_{[c,d]} \left[f(x(u))\frac{dx}{du}\right] du$$

$$= \int_{[c,d]} f(x(u))\left|\frac{dx}{du}\right| du$$

If $\frac{dx}{du} < 0$, then $d = u(b) < u(a) = c$

$$\therefore \int_a^b f(x)dx = \int_c^d \left[f(x(u))\frac{dx}{du}\right] du = -\int_{[d,c]} f(x(u))\frac{dx}{du}\, du$$

$$= \int_{[d,c]} f(x(u))\left|\frac{dx}{du}\right| du$$

Hence (in Riemann sum)

$$\int_{[a,b]} f(x)dx = \int_{[c,d]} f(x)\left|\frac{dx}{du}\right| du$$

interpreted as a set without direction

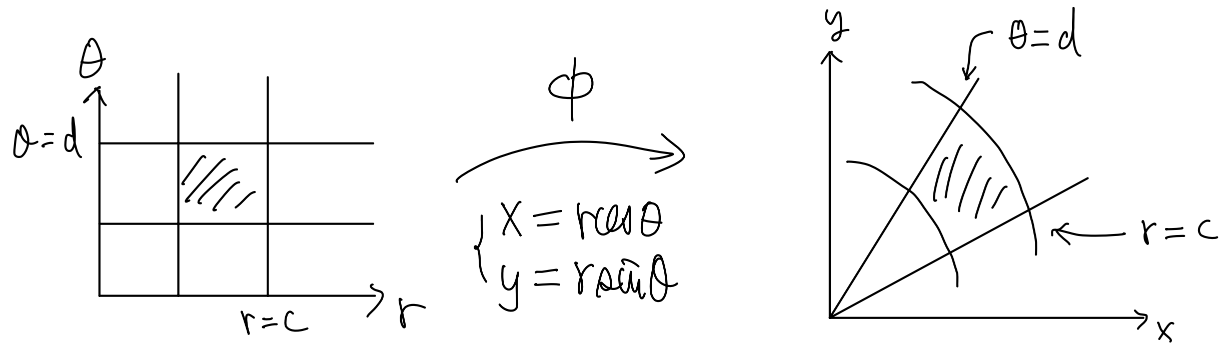
(i.e. $\{u : u$ between $c$ & $d$ (inclusive) $\}$)

& $\left|\frac{dx}{du}\right| \sim \frac{|\Delta x|}{|\Delta u|}$ ratio of lengths (of the coordinates)

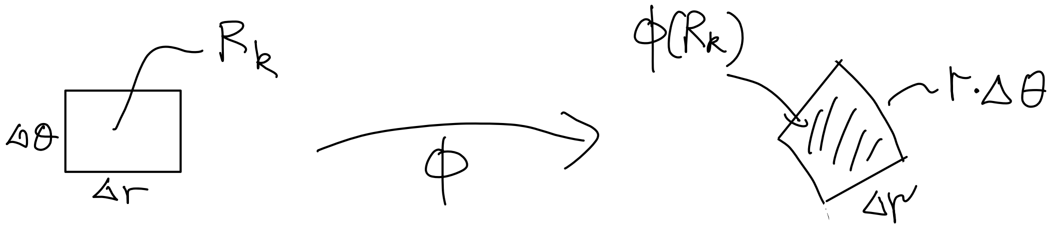
1-dim'l

# Generalization to multiple integrals

<u>eg</u>: Polar coordinates: $\displaystyle\iint_{(x,y)} f(x,y)\,dx\,dy = \iint_{(r,\theta)} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$




$\phi(r,\theta) = (x,y)$




$\text{Area}(R_k) \cong \Delta r \Delta\theta$ $\qquad$ $\text{Area}(\phi(R_k)) \cong r \Delta r \Delta\theta$

Hence

$$\frac{\text{Area}(\phi(R_k))}{\text{Area}(R_k)} \to r \quad \text{as "} R_k \to \text{point"}$$

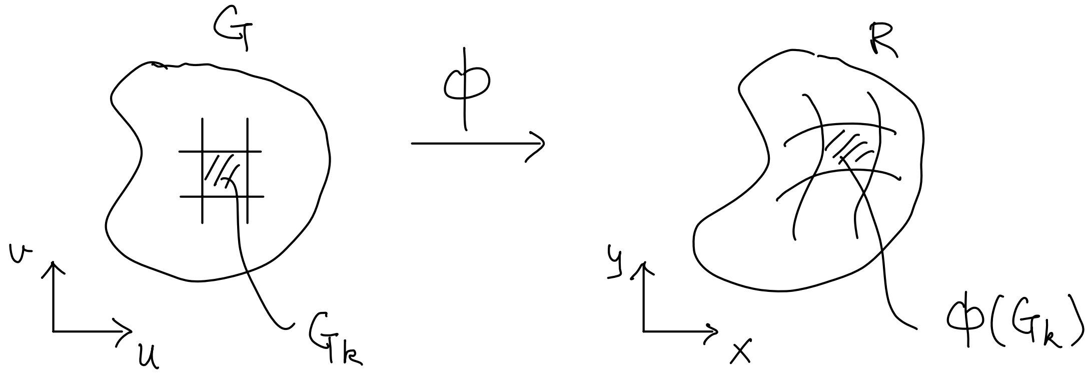
( ratio of <u>areas</u>, <u>always $\geq 0$</u> )

2-dim'l

General change of coordinate formula in $\mathbb{R}^2$

Suppose $\begin{cases} x = g(u,v) \\ y = h(u,v) \end{cases}$ is denoted by $\phi(u,v) = (x,y)$,

$$\phi : \underset{\substack{\cap \\ uv\text{-plane}}}{G} \longrightarrow \underset{\substack{\cap \\ xy\text{-plane}}}{R}$$




Idea: We need to find

$$\frac{\text{Area}(\phi(G_k))}{\text{Area}(G_k)} \longrightarrow ? \quad \text{as "} G_k \to \text{point"}$$

Assume $\phi$ is a diffeomorphism : 1-1, onto & $\phi, \phi^{-1} \in C^1$.

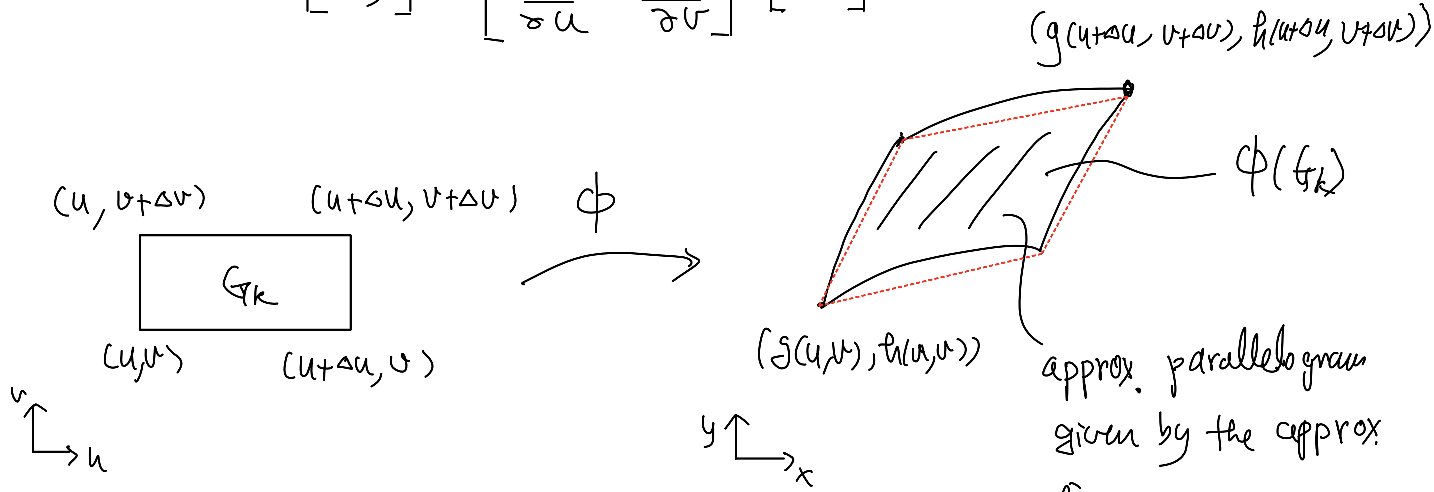
$\phi$ is $C^1 \Rightarrow$

$$\begin{cases} g(u+\Delta u, v+\Delta v) = g(u,v) + \dfrac{\partial g}{\partial u}\Delta u + \dfrac{\partial g}{\partial v}\Delta v + \cdots \\ h(u+\Delta u, v+\Delta v) = h(u,v) + \dfrac{\partial h}{\partial u}\Delta u + \dfrac{\partial h}{\partial v}\Delta v + \cdots \end{cases}$$

$$\Rightarrow \begin{cases} \Delta x = \Delta g = g(u+\Delta u, v+\Delta v) - g(u,v) = \dfrac{\partial g}{\partial u}\Delta u + \dfrac{\partial g}{\partial v}\Delta v + \cdots \\ \Delta y = \Delta h = h(u+\Delta u, v+\Delta v) - h(u,v) = \dfrac{\partial h}{\partial u}\Delta u + \dfrac{\partial h}{\partial v}\Delta v + \cdots \end{cases}$$

In matrix form

$$\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix} + \cdots$$




approx. parallelogram given by the approx linear map

(By linear algebra)

$$\frac{dA_{(x,y)}}{dA_{(u,v)}} \cong \frac{\text{Area}(\phi(G_k))}{\text{Area}(G_k)} \cong \left|\det \begin{bmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} \end{bmatrix}\right|$$

$$\cong$$

$$\left(\frac{\text{"}\Delta x \Delta y\text{"}}{\text{"}\Delta u \Delta v\text{"}}\right) = \left|\det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}\right|$$

<u>Def 7</u>: Define the <u>Jacobian</u> $J(u,v)$ of the "coordinates transformation"
$$\begin{cases} x = g(u,v) \\ y = h(u,v) \end{cases}$$
by
$$J(u,v) \stackrel{\text{notation}}{=\!=} \frac{\partial(x,y)}{\partial(u,v)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

With this notation, we should have the formula

$$\boxed{\begin{aligned}\iint\limits_{\substack{R\\ ((x,y)\in)}} f(x,y)\,dx\,dy &= \iint\limits_{\substack{G\\ ((u,v)\in)}} f(g(u,v),h(u,v))\left|\det\begin{bmatrix}\frac{\partial g}{\partial u} & \frac{\partial g}{\partial v}\\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v}\end{bmatrix}\right| du\,dv\\ &= \iint\limits_{G} f(x(u,v),y(u,v))\,|J(u,v)|\,du\,dv\\ &= \iint\limits_{G} f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv\end{aligned}}$$

eg 28: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ $\quad ((u,v)=(r,\theta))$

$$\Rightarrow \quad J(r,\theta) = \frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{bmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{bmatrix} = r \qquad \text{(check!)}$$

and
$$\iint\limits_{R} f(x,y)\,dx\,dy = \iint\limits_{G} f(r\cos\theta, r\sin\theta)\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta$$
$$= \iint\limits_{G} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

(same formula as before.)

**Thm 6**: Suppose $\phi : \begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix}$ is a diffeomorphism (1-1, onto, s.t. $\phi$ and $\phi^{-1} \in C^1$) mapping a region $G$ (closed and bounded) in the $uv$-plane onto a region $R$ (closed and bounded) in the $xy$-plane (except possibly on the boundary).

Suppose $f(x,y)$ is <u>continuous</u> on $R$, then

$$\iint_R f(x,y)\,dx\,dy = \iint_G f\circ\phi(u,v) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

<u>Notes</u>: (i) $f\circ\phi(u,v) = f(x(u,v), y(u,v))$

(ii) $\phi$ is a diffeomorphism $\Rightarrow \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \neq 0$.

<u>Triple integrals</u> ("substitutions" in triple integrals)

$$\phi(u,v,w) = (x,y,z) : G^{\subset \mathbb{R}^3 \ni (u,v,w)} \longrightarrow D^{\subset \mathbb{R}^3 \ni (x,y,z)}$$

with
$$\begin{cases} x = g(u,v,w) \\ y = h(u,v,w) \\ z = k(u,v,w) \end{cases}$$
1-1, onto, cont. differentiable and <u>inverse</u> also cont. differentiable.

<u>Def 8</u> <u>Jacobian</u> (<u>determinant</u>) of transformation in $\mathbb{R}^3$

$$J(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix} \left( = \det \begin{bmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} & \frac{\partial g}{\partial w} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} & \frac{\partial h}{\partial w} \\ \frac{\partial k}{\partial u} & \frac{\partial k}{\partial v} & \frac{\partial k}{\partial w} \end{bmatrix} \right)$$

Note: Chain rule $\Rightarrow$

$$\begin{cases} \text{2-dim} & \dfrac{\partial(x,y)}{\partial(u,v)} \cdot \dfrac{\partial(u,v)}{\partial(s,t)} = \dfrac{\partial(x,y)}{\partial(s,t)} \\[2ex] \text{3-dim} & \dfrac{\partial(x,y,z)}{\partial(u,v,w)} \cdot \dfrac{\partial(u,v,w)}{\partial(s,t,r)} = \dfrac{\partial(x,y,z)}{\partial(s,t,r)} \end{cases}$$

(Ex!)

$$\Rightarrow \begin{cases} \text{2-dim} & \dfrac{\partial(u,v)}{\partial(x,y)} = \dfrac{1}{\dfrac{\partial(x,y)}{\partial(u,v)}} \\[3ex] \text{3-dim} & \dfrac{\partial(u,v,w)}{\partial(x,y,z)} = \dfrac{1}{\dfrac{\partial(x,y,z)}{\partial(u,v,w)}} \end{cases}$$

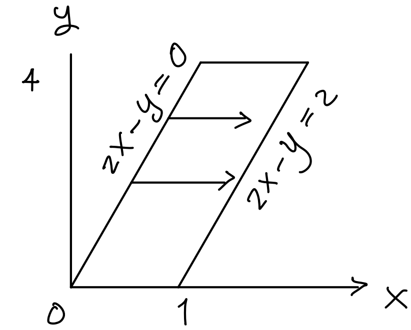
(Ex!)

**Thm 7:** Under similar conditions of Thm 6

$$\iiint_D F(x,y,z)\,dx\,dy\,dz = \iiint_G F\circ\phi(u,v,w)\,\left|J(u,v,w)\right|\,du\,dv\,dw$$

$$= \iiint_G F\big(g(u,v,w),\, h(u,v,w),\, k(u,v,w)\big)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

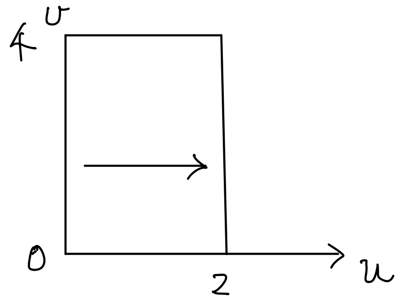
eg 29
$$\int_0^4 \int_{\frac{y}{2}}^{\frac{y}{2}+1} \frac{2x-y}{2}\,dx\,dy$$




Soln lower limit $x = \frac{y}{2} \Leftrightarrow 2x-y=0$

upper limit $x = \frac{y}{2}+1 \Leftrightarrow 2x-y=2$

Define $\begin{cases} u = 2x - y \\ v = y \end{cases}$




Then $\begin{cases} x = \frac{1}{2}u + \frac{1}{2}v \\ y = v \end{cases}$

$\begin{cases} 2x - y = 0 \\ 2x - y = 2 \end{cases} \longleftrightarrow \begin{matrix} u = 0 \\ u = 2 \end{matrix}$ $\qquad$ $\begin{cases} y = 0 \\ y = 4 \end{cases} \longleftrightarrow \begin{matrix} v = 0 \\ v = 4 \end{matrix}$

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \det\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix} = \frac{1}{2}$$

$$\therefore \int_0^4 \int_{\frac{y}{2}}^{\frac{y}{2}+1} \frac{2x-y}{2} \, dx\, dy = \int_0^4 \int_0^2 \frac{u}{2} \left|\frac{1}{2}\right| du\, dv = 2 \quad \text{(Check!)}$$

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