Double integral of f over $R = \{(r,\theta) : a \le r \le b,\ c \le \theta \le d\}$ in polar coordinates is

$$\iint_R f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta = \int_c^d \left(\int_a^b f(r,\theta)\, r\,dr \right) d\theta$$

$$= \int_a^b \left(\int_c^d f(r,\theta)\, d\theta \right) r\,dr$$

where $f(r,\theta)$ is the simplified notation for $f(r\cos\theta, r\sin\theta)$

Remark: This is a special case of the change of variables formula. The "extra" factor "r" in the integrand is in fact

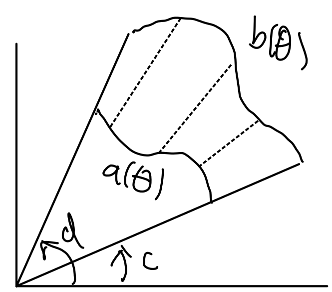
$$r = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$ the Jacobian determinant of the change of variables.

More generally

Thm 3: If R is a (closed and bounded) region with

$a(\theta) \le r \le b(\theta)$ and $c \le \theta \le d$

$(0 \le a(\theta) \le b(\theta),\ a(\theta) \not\equiv b(\theta))$

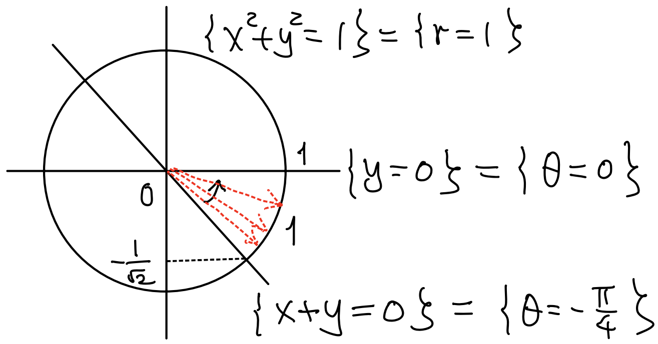


And $f : R \to \mathbb{R}$, then

$$\iint_R f(x,y)\,dA = \int_c^d \left(\int_{a(\theta)}^{b(\theta)} f(r\cos\theta, r\sin\theta)\, r\,dr \right) d\theta$$

(remember this extra "r")

eg 12: Back to our previous example 9



$$R = \{ 0 \le r \le 1, -\frac{\pi}{4} \le \theta \le 0 \}$$

$$f(x,y) = x = r\cos\theta$$

$$\iint_R x \, dA = \int_{-\frac{\pi}{4}}^{0} \left(\int_0^1 r\cos\theta \cdot r\,dr \right) d\theta = \left(\int_{-\frac{\pi}{4}}^{0} \cos\theta \, d\theta \right) \left(\int_0^1 r^2 dr \right)$$

$$= \cdots = \frac{1}{3\sqrt{2}} \quad (\text{check!})$$

Much easier than before!

eg 13 Convert integrals between Cartestan and Polar coordinates

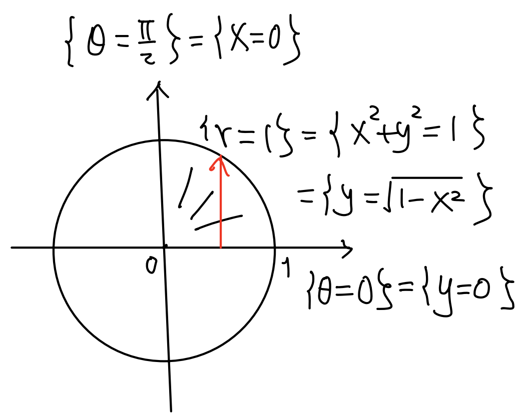
(a) $\int_0^{\frac{\pi}{2}} \int_0^1 r^3 \sin\theta \cos\theta \, dr \, d\theta$

(b) $\int_1^2 \int_0^{\sqrt{2x-x^2}} y \, dy \, dx$

Soln: (a) $\int_0^{\frac{\pi}{2}} \int_0^1 r^3 \sin\theta\cos\theta \, dr \, d\theta$

$$= \int_0^{\frac{\pi}{2}} \left(\int_0^1 (r\cos\theta)(r\sin\theta) \, r\,dr \right) d\theta$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} xy \, dy \, dx$$

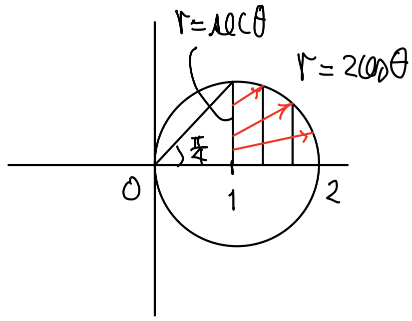


(b)

$$\int_1^2 \int_0^{\sqrt{2x-x^2}} y\,dy\,dx$$

$$= \int_0^{\frac{\pi}{4}} \left(\int_{\sec\theta}^{2\cos\theta} r\sin\theta \cdot r\,dr \right) d\theta$$

$$= \int_0^{\frac{\pi}{4}} \int_{\sec\theta}^{2\cos\theta} r^2 \sin\theta\,dr\,d\theta$$



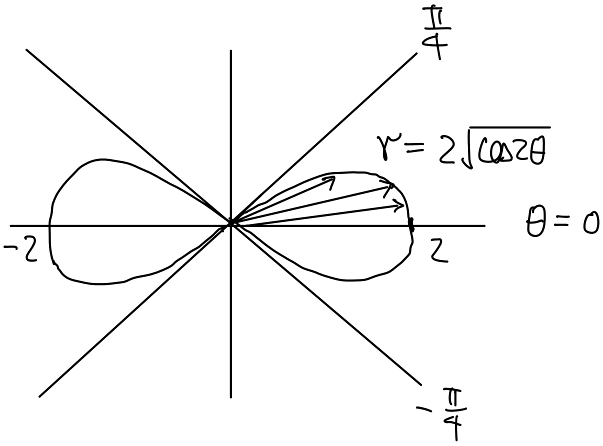
eg 14: Find area enclosed by $r^2 = 4\cos 2\theta$.

Soln: By symmetry

$$\text{Area} = 4 \int_0^{\frac{\pi}{4}} \left[\int_0^{2\sqrt{\cos 2\theta}} 1 \cdot r\,dr \right] d\theta$$

$$= 4 \int_0^{\frac{\pi}{4}} \left[\frac{r^2}{2} \right]_0^{2\sqrt{\cos 2\theta}} d\theta$$

$$= 8 \int_0^{\frac{\pi}{4}} \cos 2\theta \, d\theta = 4 \quad \text{(check!)}$$



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Remark: r is "not really" a function of all θ, it should be regarded as a "level set":

(i) there is no solution when $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$ & $\frac{5\pi}{4} < \theta < \frac{7\pi}{4}$

(ii) in term of (x, y) coordinates

$$F(x,y) = (x^2+y^2)^2 - 4(x^2-y^2) = 0 \quad \text{(check!)}$$

which has a critical point at $(x,y)=(0,0)$ $\left(\vec{\nabla}F(0,0)=\vec{0}\right)$

on the level set. (One cannot apply "Implicit Function Theorem"

at the critical point $(0,0)$) (later for more detail)

eg15: Integrate $f(x,y)=\dfrac{1}{\sqrt{x^2+y^2}}$ over the region R bounded

between

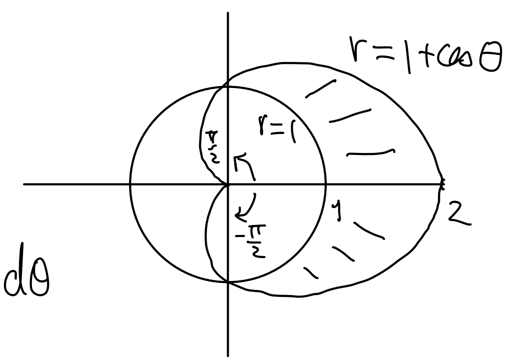
$$\begin{cases} r = 1+\cos\theta & \text{(cardiod)} \\ r = 1 \end{cases}$$

and <u>outside</u> the circle $r=1$.

<u>Soln</u> $r=1+\cos\theta$ & $r=1$

Intersection : $1 = 1+\cos\theta$

$\Rightarrow \theta = \pm\frac{\pi}{2}$



$$\iint_R f(x,y)\,dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\int_1^{1+\cos\theta} \frac{1}{r}\cdot r\,dr \right] d\theta$$

$= \cdots = 2$ (Check!)

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