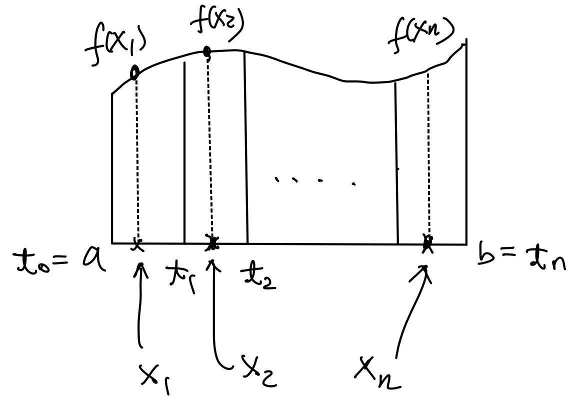
Double Integrals

Recall: In one-variable, "integral" is regarded as "limit" of "Riemann sum" (take MATH 2060 for rigorous treatment)

$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k)\Delta x_k$$

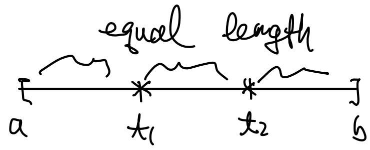
where
- f is a function on the interval $[a,b]$
- P is a partition $a = t_0 < t_1 < t_2 < \cdots < t_n = b$
- $x_k \in [t_{k-1}, t_k]$ and $\Delta x_k = t_k - t_{k-1}$
- $\|P\| = \max_k |\Delta x_k|$



Remark: We usually use uniform partition P

$$a = t_0 < t_1 = a + \frac{1}{n}(b-a) < t_2 = a + \frac{2}{n}(b-a) < \cdots$$

$$\cdots < t_k = a + \frac{k}{n}(b-a) < \cdots = t_n = b$$



In this case, $\|P\| = \max_k |\Delta x_k| = \dfrac{b-a}{n} \to 0$ as $n \to \infty$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x_k = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\cdot\frac{b-a}{n}$$

eg1: Find $\int_0^1 x^3 dx$ (i.e. $f(x)=x^3$ on $[0,1]$)

Soln: (1) One may choose $x_k = \frac{k-1}{n} \in [\frac{k-1}{n}, \frac{k}{n}]$

then $S_n = \sum_{k=1}^{n} f(x_k)\Delta x_k = \sum_{k=1}^{n} \left(\frac{k-1}{n}\right)^3 \frac{1}{n}$

$= \cdots = \frac{1}{4}\left(1-\frac{1}{n}\right)^2$ (Ex!)

$\to \frac{1}{4}$ as $n \to \infty$

$\therefore \int_0^1 x^3 dx = \frac{1}{4}$

(2) Or, we can choose $x_k = \frac{k}{n} \in [\frac{k-1}{n}, \frac{k}{n}]$

then $S_n = \sum_{k=1}^{n} \left(\frac{k}{n}\right)^3 \frac{1}{n} = \cdots = \frac{1}{4}\left(1+\frac{1}{n}\right)^2 \to \frac{1}{4}$ as $n \to \infty$

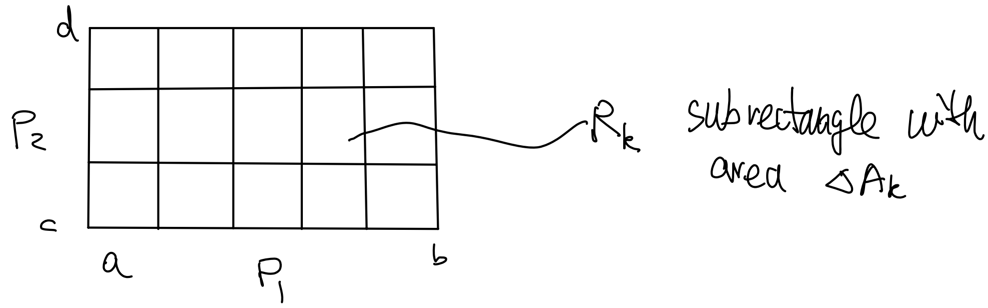
(Ex!)

$\therefore \int_0^1 x^3 dx = \frac{1}{4}$

Remark: We can use any $x_k \in [t_{k-1}, t_k]$ and still get the same $\int_0^1 x^3 dx = \frac{1}{4}$.

This concept can be generalized to any dimension.

For 2-dim., let we first consider a function $f(x,y)$ defined on a (closed) rectangle $R = [a,b] \times [c,d] = \{(x,y): a \le x \le b,\ c \le y \le d\}$



Then we can subdivide R into sub-rectangles by using partitions P_1 of $[a,b]$ & P_2 of $[c,d]$.

Denote $P = P_1 \times P_2$ (partition, subdivision, of R)

and $\|P\| = \max(\|P_1\|, \|P_2\|)$

Let the sub-rectangles be R_k, $k=1,\cdots,N$ (= number of sub rectangles)

with <u>areas</u> ΔA_k

Choose point $(x_k, y_k) \in R_k$ (for each $k=1,\cdots,N$),

then consider the sum

$$S(f,P) = \sum_{k=1}^{N} f(x_k,y_k)\,\Delta A_k$$

<u>Def 1</u>: The function f is said to be <u>integrable</u> over R

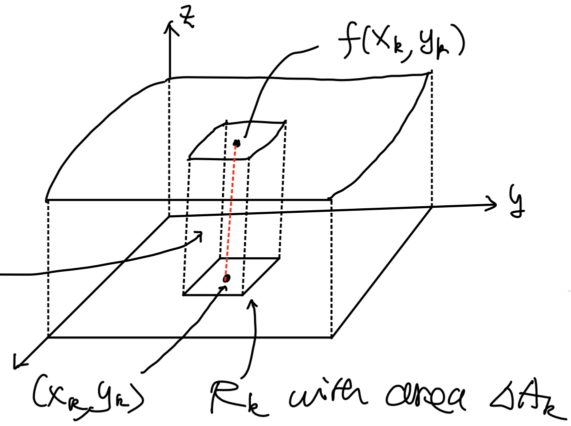
if $\displaystyle\lim_{\|P\|\to 0} S(f,P) = \lim_{\|P\|\to 0} \sum_{k=1}^{N} f(x_k,y_k)\,\Delta A_k$

<u>exists</u> and <u>independent</u> of the choice of $(x_k,y_k) \in R_k$.

In this case, the limit is called the <u>(double) integral</u> <u>of f over R</u> and is denoted by

$$\iint_R f(x,y)\,dA \quad \text{or} \quad \iint_R f(x,y)\,dx\,dy$$

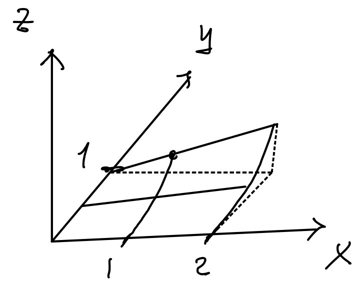
Remarks: (1) Same as 1-variable, the double integral of f, $f \geq 0$, over R can be interpreted as volume under the graph of f (over R)



(2) And when $f \equiv 1$,

$$\iint_R 1 \, dA = \text{the area of } R.$$

eg 2: $R = [0,2] \times [0,1]$, $f(x,y) = xy^2$
(using definition) Find $\iint_R xy^2 \, dx\,dy$



Soln: Using uniform partitions:

$$P_1 = \{0, \tfrac{2}{n}, \tfrac{4}{n}, \cdots, 2\} \text{ of } [0,2]$$

$$P_2 = \{0, \tfrac{1}{n}, \tfrac{2}{n}, \cdots, 1\} \text{ of } [0,1]$$

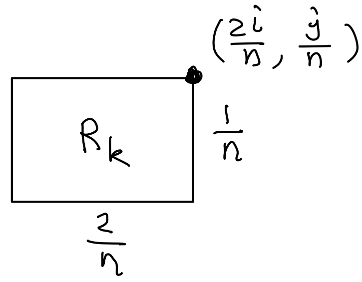
$\Rightarrow$ a particular sub-rectangle is

$$R_k = [\tfrac{2}{n}(i-1), \tfrac{2}{n}i] \times [\tfrac{1}{n}(j-1), \tfrac{1}{n}j] \quad \text{for} \quad \begin{matrix} i = 1, \cdots, n \\ j = 1, \cdots, n \end{matrix}$$

(it is better denoted by R_{ij})

One may choose the point

$$(x_k, y_k) = \left(\frac{2i}{n}, \frac{j}{n}\right) \in R_k$$



and consider the Riemann Sum

$$\sum_k f(x_k, y_k)\,\Delta A_k = \sum_{i,j=1}^{n} \left(\frac{2i}{n}\right)\left(\frac{j}{n}\right)^2 \cdot \frac{2}{n}\frac{1}{n}$$

$$= \frac{4}{n^5}\sum_{i,j=1}^{n} i j^2 = \frac{4}{n^5}\sum_{i=1}^{n}\left(\sum_{j=1}^{n} i j^2\right)$$

$$= \frac{4}{n^5}\sum_{i=1}^{n}\left(i\sum_{j=1}^{n} j^2\right) = \frac{4}{n^5}\left(\sum_{i=1}^{n} i\right)\left(\sum_{j=1}^{n} j^2\right)$$

$$= \frac{4}{n^5}\cdot\frac{n(n+1)}{2}\cdot\frac{n(n+1)(2n+1)}{6}$$

$$\longrightarrow \frac{2}{3} \quad \text{as } n \to \infty$$

$$\therefore \iint_{[0,2]\times[0,1]} xy^2\,dx\,dy = \frac{2}{3} \quad \#$$

Very tedious calculation.

Hence we need the following Theorem:

<u>Thm 1</u> (<u>Fubini's Theorem</u> (<u>1st form</u>))

If $f(x,y)$ is <u>continuous</u> on $R = [a,b]\times[c,d]$, then

$$\iint_R f(x,y)\,dA = \int_c^d\left[\int_a^b f(x,y)\,dx\right]dy = \int_a^b\left[\int_c^d f(x,y)\,dy\right]dx$$

The last 2 integrals above are called <u>iterated integrals</u>

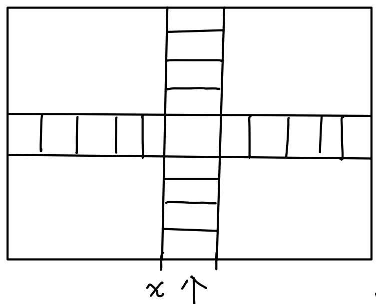
(Pf: Omitted)

<u>Ideas</u>:

sum horizontally first & taking limit

$\to \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy$

(" $\sum_i$ " 1st in eg2)



sum vertically first & taking limit

$\to \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$ (" $\sum_j$ " 1st in eg2)

<u>eg3</u>: Using Fubini to calculate $\iint_R xy^2\,dxdy$, $R=[0,2]\times[0,1]$

<u>Sol</u>: By Fubini,

$$\iint_R xy^2\,dxdy = \int_0^2\left(\int_0^1 xy^2\,dy\right)dx$$

$$= \int_0^2\left(x\int_0^1 y^2\,dy\right)dx$$

$$= \left(\int_0^2 x\,dx\right)\left(\int_0^1 y^2\,dy\right)$$

$$= \frac{2}{3}$$

(Much easier than eg2)

<u>eg4</u>: Some times the "order" of the iterated integrals is important in practice:

Find $\iint_{[0,1]\times[0,\pi]} x\sin(xy)\,dA$.

<u>Soln</u>: $\iint_{[0,1]\times[0,\pi]} x\sin(xy)\,dA = \int_0^{\pi}\left[\int_0^1 x\sin(xy)\,dx\right]dy$

$= \int_0^{\pi}\left(-\frac{\cos y}{y} + \frac{\sin y}{y^2}\right)dy$ (integration-by-parts)

Not easy to integrate!

On the other hand,

$\iint_{[0,1]\times[0,\pi]} x\sin(xy)\,dA = \int_0^1\left[\int_0^{\pi} x\sin(xy)\,dy\right]dx$

$= \int_0^1 (-\cos\pi x + 1)\,dx$

$= 1$ (easy!) #

<u>Caution</u>: <u>Not</u> all functions are integrable over a (closed) rectangle.

<u>Remark</u>: • To show "integrable", needs to show that <u>for all partitions</u> and <u>for all points (x_k, y_k)</u> in the subrectangles, the Riemann sum

$S(f,P) \to$ the <u>same number</u> (as $\|P\|\to 0$)

- To <u>disprove</u> "integrable", needs to find, for examples

(i) <u>some</u> P with <u>some</u> choose of (x_k, y_k) such that

$$\lim_{\|P\| \to 0} S(f, P) \text{ doesn't exist.}$$

(ii) <u>some</u> P with <u>different</u> (x_k, y_k) & (x_k', y_k') such that

$$\underset{\text{with } (x_k, y_k)}{S(f, P)} \to a \neq b \leftarrow \underset{\text{with } (x_k', y_k')}{S'(f, P)}$$