15.5

Surface Area of Parametrized Surfaces

In Exercises 17–26, use a parametrization to express the area of the surface as a double integral. Then evaluate the integral. (There are many correct ways to set up the integrals, so your integrals may not be the same as those in the back of the text. They should have the same values, however.)

20. Cone frustum The portion of the cone $z = \sqrt{x^2+y^2}/3$ between the planes $z = 1$ and $z = 4/3$

Sol'n: Use cylindrical coordinates:

$x = r\cos\theta$
$y = r\sin\theta$
$z = \dfrac{\sqrt{x^2+y^2}}{3} = \dfrac{r}{3}$

$0 \le \theta \le 2\pi.$

$1 \le z \le \frac{4}{3}$
$\Updownarrow$
$3 \le r \le 4.$

Then parameterization is given by

$\vec{r}(r,\theta) = r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j} + \frac{r}{3}\hat{k}$

$\vec{r}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} + \frac{1}{3}\hat{k}$

$\vec{r}_\theta = -r\sin\theta\,\hat{i} + r\cos\theta\,\hat{j} + 0\hat{k}$

$$A = \int_0^{2\pi}\int_3^4 |\vec{r}_r \times \vec{r}_\theta|\,dr\,d\theta.$$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & \frac{1}{3} \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = -\frac{1}{3} r\cos\theta\,\hat{i} - \frac{1}{3} r\sin\theta\,\hat{j} + r\hat{k}$$

So $|\vec{r}_r \times \vec{r}_\theta| = \left(\left(-\frac{1}{3} r\cos\theta\right)^2 + \left(-\frac{1}{3} r\sin\theta\right)^2 + r^2\right)^{1/2} = \left(\frac{1}{9}r^2 + r^2\right)^{\frac{1}{2}} = \sqrt{\frac{10}{9}r^2} = r\frac{\sqrt{10}}{3}$

So $A = \int_0^{2\pi}\int_3^4 r\frac{\sqrt{10}}{3}\,dr\,d\theta = \frac{2\pi\sqrt{10}}{3}\cdot\frac{1}{2}r^2\Big|_3^4 = \boxed{\frac{7\sqrt{10}\pi}{3}}$

15.3

24. Parabolic band The portion of the paraboloid $z = x^2 + y^2$ between the planes $z = 1$ and $z = 4$

Sol'n: $x = r\cos\theta$ $\quad 0 \le \theta \le 2\pi$

$y = r\sin\theta$

$z = r^2$ $\quad 1 \le z \le 4 \Rightarrow 1 \le r^2 \le 4 \Rightarrow 1 \le r \le 2.$

$$\vec{r}(r,\theta) = r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j} + r^2\hat{k}$$

$$\vec{r}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} + 2r\hat{k}$$

$$\vec{r}_\theta = -r\sin\theta\,\hat{i} + r\cos\theta\,\hat{j} + 0\hat{k}$$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = -2r^2\cos\theta\,\hat{i} - 2r^2\sin\theta\,\hat{j} + r\hat{k}$$

$$|\vec{r}_r \times \vec{r}_\theta| = \left((-2r^2\cos\theta)^2 + (-2r^2\sin\theta)^2 + r^2\right)^{1/2} = \sqrt{4r^4 + r^2} = r\sqrt{4r^2+1}$$

Then $A = \int_0^{2\pi}\int_1^2 r\sqrt{4r^2+1}\,dr\,d\theta = 2\pi\int_5^{17} \frac{1}{8}\sqrt{u}\,du = \frac{\pi}{6}u^{3/2}\Big|_5^{17}$

$u = 4r^2+1$

$du = 8r\,dr$

$1 \le r \le 2 \Rightarrow 5 \le u \le 17$

$= \boxed{\frac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5}\right)}$

15.5

Planes Tangent to Parametrized Surfaces

The tangent plane at a point $P_0(f(u_0, v_0), g(u_0, v_0), h(u_0, v_0))$ on a parametrized surface $\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}$ is the plane through P_0 normal to the vector $\mathbf{r}_u(u_0, v_0) \times \mathbf{r}_v(u_0, v_0)$, the cross product of the tangent vectors $\mathbf{r}_u(u_0, v_0)$ and $\mathbf{r}_v(u_0, v_0)$ at P_0. In Exercises 27–30, find an equation for the plane tangent to the surface at P_0. Then find a Cartesian equation for the surface, and sketch the surface and tangent plane together.

30. Parabolic cylinder The parabolic cylinder surface $\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} - x^2\mathbf{k}$, $-\infty < x < \infty$, $-\infty < y < \infty$, at the point $P_0(1, 2, -1)$ corresponding to $(x, y) = (1, 2)$

Sol'n: $\vec{r}(x,y) = x\hat{i} + y\hat{j} - x^2\hat{k}$

$\vec{r}_x = \hat{i} + 0\hat{j} - 2x\hat{k}$

$\vec{r}_y = 0\hat{i} + \hat{j} + 0\hat{k}$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -2x \\ 0 & 1 & 0 \end{vmatrix} = 2x\hat{i} + 0\hat{j} + \hat{k}.$$ So $\vec{r}_x \times \vec{r}_y \big|_{P_0} = (2, 0, 1)$

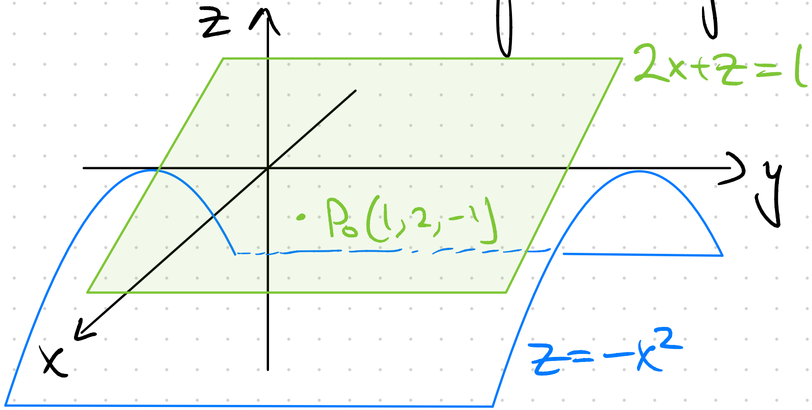
So tangent plane at $P_0(1,2,-1)$ is given by

$$2(x-1) + 0(y-2) + (z-(-1)) = 0 \iff 2x - 2 + z + 1 = 0 \iff 2x + z = 1$$

Cartesian equation: $\vec{r}(x,y) = (x, y, -x^2)$

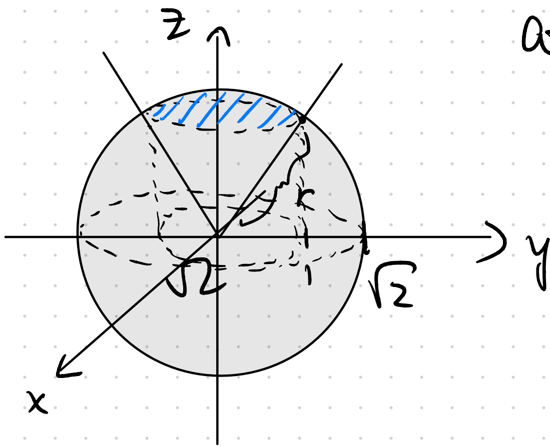
So cartesian equation given by $z = -x^2$.

Sketch:



15.5 **42.** Find the area of the cap cut from the sphere $x^2 + y^2 + z^2 = 2$ by the cone $z = \sqrt{x^2 + y^2}$.

Sol'n: Method of Level Surface.



At the intersection, (substitute $z=\sqrt{x^2+y^2}$ into $x^2+y^2+z^2=2$)

$x^2+y^2+x^2+y^2=2 \Rightarrow x^2+y^2=1$

$\Rightarrow r=1$

when $x=r\cos\theta$, $y=r\sin\theta$.

So Ω = disk of radius 1. Let $F(x,y,z) = x^2+y^2+z^2=2$.

Then $\text{Area} = \iint\limits_{\Omega} \frac{|\vec{\nabla}F|}{|F_z|}\, d\sigma =$

$\vec{\nabla}F = (2x, 2y, 2z)$ and $|\vec{\nabla}F| = \sqrt{4x^2+4y^2+4z^2}$

$= 2\sqrt{x^2+y^2+z^2} = 2\sqrt{2}$. (since $x^2+y^2+z^2=2$)

and $|F_z| = 2z \underset{\uparrow}{=} 2\left(\sqrt{2-x^2-y^2}\right) = 2\left(\sqrt{2-r^2}\right)$ when $x = r\cos\theta$

$F=2$

$y = r\sin\theta$

$0 \le r \le 1$

$0 \le \theta \le 2\pi$.

$$\text{So } A = \int_0^{2\pi}\int_0^1 \frac{2\sqrt{2}}{2\sqrt{2-r^2}}\, r\,dr\,d\theta = 2\sqrt{2}\pi \int_0^1 \frac{r}{\sqrt{2-r^2}}\,dr$$

$$= 2\sqrt{2}\pi\left(-\sqrt{2-r^2}\right)\Big|_{r=0}^{r=1}$$

$$= 2\sqrt{2}\pi\left(\sqrt{2}-1\right) = \boxed{4\pi - 2\sqrt{2}\pi}$$

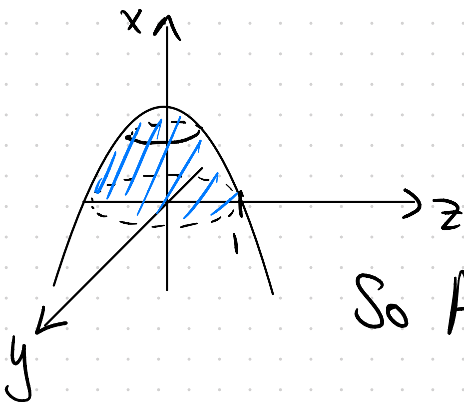
15.5

Find the area of the surfaces in Exercises 49–54.

49. The surface cut from the bottom of the paraboloid $z = x^2 + y^2$ by the plane $z = 3$

50. The surface cut from the "nose" of the paraboloid $x = 1 - y^2 - z^2$ by the yz-plane

Sol'n:



a graph given by $x = f(y,z) = 1 - y^2 - z^2$.

when $x=0$, $1 = y^2 + z^2$, so Ω = disk of radius 1.

So $A = \iint_\Omega \sqrt{1 + |\vec{\nabla} f|}\, d\sigma$

$\vec{\nabla} f = (-2y, -2z)$ so $|\vec{\nabla} f| = 4y^2 + 4z^2 = 4(y^2+z^2)$.

Taking polar coordinates $y = r\cos\theta$, $z = r\sin\theta$, $y^2+z^2 = r^2$,

$0 \le r \le 1$, $0 \le \theta \le 2\pi$.

So $A = \iint_\Omega \sqrt{1 + |\vec{\nabla} f|}\, d\sigma = \int_0^{2\pi}\int_0^1 \sqrt{1+4r^2}\, r\,dr\,d\theta$

$$= \int_0^{2\pi} \frac{1}{12}(1+4r^2)^{3/2}\Big|_0^1 \, d\theta$$

$$= \boxed{\frac{\pi}{6}\left(5\sqrt{5}-1\right)}$$

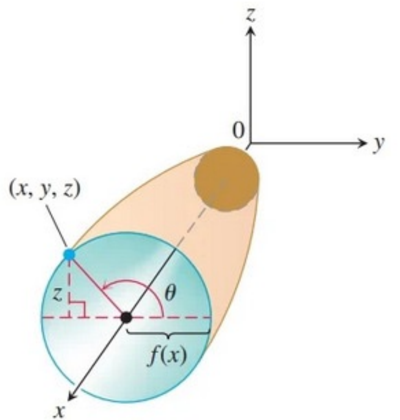
15.5

56. Let S be the surface obtained by rotating the smooth curve $y = f(x)$, $a \le x \le b$, about the x-axis, where $f(x) \ge 0$.

a. Show that the vector function

$$\mathbf{r}(x, \theta) = x\mathbf{i} + f(x)\cos\theta\,\mathbf{j} + f(x)\sin\theta\,\mathbf{k}$$

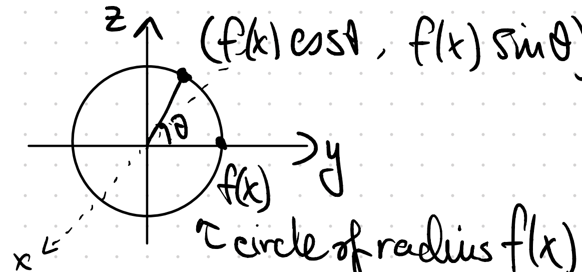
is a parametrization of S, where θ is the angle of rotation around the x-axis (see the accompanying figure).



b. Use Equation (4) to show that the surface area of this surface of revolution is given by

$$A = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\,dx.$$

a) Cross section at fixed $x \in [a,b]$.



← circle of radius $f(x)$.

So any point on the surface of revolution at fixed x is given by $\vec{r}(x,\theta) = x\hat{i} + f(x)\cos\theta\,\hat{j} + f(x)\sin\theta\,\hat{k}$, as required.

b) Equation (4) refers to

$$A = \iint_R |\vec{r}_x \times \vec{r}_\theta|\,dA$$

In our case, $a \le x \le b$, $0 \le \theta \le 2\pi$.

and $\vec{r}_x = \hat{i} + f'(x)\cos\theta\,\hat{j} + f'(x)\sin\theta\,\hat{k}$

$\vec{r}_\theta = 0\hat{i} - f(x)\sin\theta\,\hat{j} + f(x)\cos\theta\,\hat{k}$.

$$\vec{r}_x \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & f'(x)\cos\theta & f'(x)\sin\theta \\ 0 & -f(x)\sin\theta & f(x)\cos\theta \end{vmatrix}$$

$$= \left(f'(x)f(x)\cos^2\theta + f'(x)f(x)\sin^2\theta\right)\hat{i} + f(x)\cos\theta\,\hat{j} - f(x)\sin\theta\,\hat{k}$$

$$= f'(x)f(x)\hat{i} + f(x)\cos\theta\,\hat{j} - f(x)\sin\theta\,\hat{k}.$$

and $|\vec{r}_x \times \vec{r}_\theta| = \left(\left(f'(x)f(x)\right)^2 + (f(x))^2\cos^2\theta + (f(x))^2\sin^2\theta\right)^{1/2}$

$$= \sqrt{f(x)^2\left(1+\left[f'(x)\right]^2\right)}$$

So $A = \int_a^b \int_0^{2\pi} \sqrt{f(x)^2\left(1+(f'(x))^2\right)}\,d\theta\,dx = \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$ as required

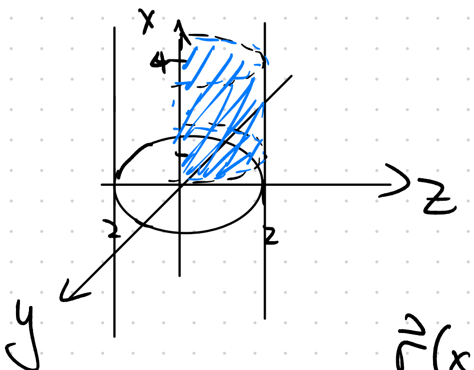
15.6

Surface Integrals of Scalar Functions

In Exercises 1–8, integrate the given function over the given surface.

1. **Parabolic cylinder** $G(x, y, z) = x$, over the parabolic cylinder $y = x^2$, $0 \le x \le 2$, $0 \le z \le 3$
2. **Circular cylinder** $G(x, y, z) = z$, over the cylindrical surface $y^2 + z^2 = 4$, $z \ge 0$, $1 \le x \le 4$

Soln:



Parameterization of cylindrical surface,

$y = 2\cos\theta$ $\quad 0 \le \theta \le \pi \quad (z \ge 0)$.

$z = 2\sin\theta$,

$x = x$. $\quad 1 \le x \le 4$

$$\vec{r}(x,\theta) = x\hat{i} + 2\cos\theta\hat{j} + 2\sin\theta\hat{k}.$$

Then $\displaystyle\iint_S G(x,y,z)\,d\sigma = \int_0^{\pi}\int_1^4 G(\vec{r}(x,\theta))\,|\vec{r}_x \times \vec{r}_\theta|\,dx\,d\theta$.

$G(\vec{r}(x,\theta)) = 2\sin\theta$.

$\vec{r}_x = \hat{i} + 0\hat{j} + 0\hat{k}$. $\quad \vec{r}_\theta = 0\hat{i} - 2\sin\theta\hat{j} + 2\cos\theta\hat{k}$.

$$\vec{r}_x \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & -2\sin\theta & 2\cos\theta \end{vmatrix} = 0\hat{i} + 2\cos\theta\,\hat{j} - 2\sin\theta\,\hat{k}$$

and $|\vec{r}_x \times \vec{r}_\theta| = (4\cos^2\theta + 4\sin^2\theta)^{1/2} = 2.$

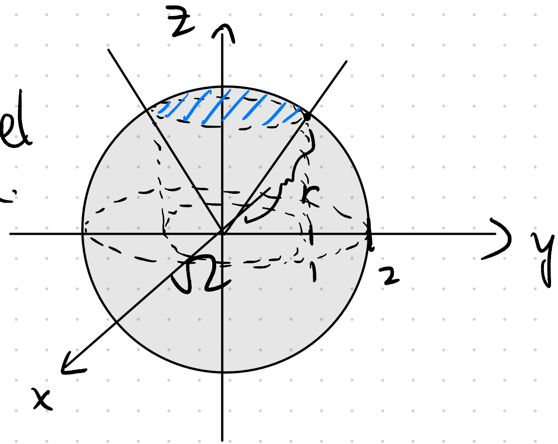
So $$\int_0^{\pi}\int_1^4 G(\vec{r}(x,\theta))\,|\vec{r}_x \times \vec{r}_\theta|\,dx\,d\theta = \int_0^{\pi}\int_1^4 4\sin\theta\,dx\,d\theta$$

$$= \int_0^{\pi} 4x\sin\theta\Big|_1^4 \,dx\,d\theta = \int_0^{\pi} 12\sin\theta\,d\theta = -12\cos\theta\Big|_0^{\pi} = 12-(-12) = \boxed{24}.$$

15.6 **8. Spherical cap** $H(x, y, z) = yz$, over the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the cone $z = \sqrt{x^2 + y^2}$

Sol'n:

again use level set method.



$$x^2+y^2+x^2+y^2=4 \Rightarrow x^2+y^2=2.$$ So Ω is disk of radius $\sqrt{2}$.

$$F(x,y,z) = x^2+y^2+z^2.$$

$$\vec{\nabla}F = (2x, 2y, 2z), \quad |\vec{\nabla}F| = (4x^2+4y^2+4z^2)^{1/2} = (4(x^2+y^2+z^2))^{1/2}$$

$$= \sqrt{4 \cdot 4} = 4.$$

$$|F_z| = |2z|.$$

So $\displaystyle\iint_S H\,d\sigma = \iint_{(x,y)\in\Omega} H(x,y,z)\,\frac{4}{2z}\,dx\,dy$

$x = r\cos\theta$, $y = r\sin\theta$, $0\le\theta\le 2\pi$, $0\le r\le\sqrt{2}$

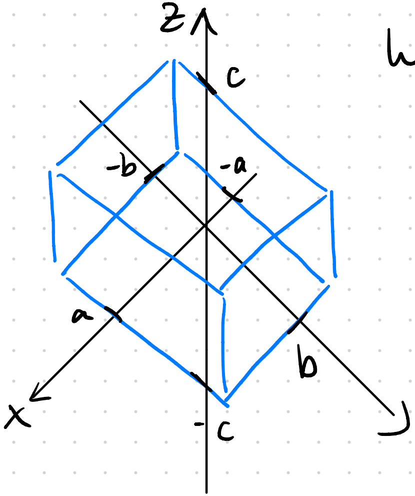
$$= \iint_{(x,y)\in\Omega} y\cancel{z}\,\frac{\cancel{4}^{\,2}}{\cancel{2z}}\,dx\,dy = \iint_{(x,y)\in\Omega} 2y\,dx\,dy$$

$$= \int_0^{2\pi}\int_0^{\sqrt{2}} 2r^2\sin\theta\,dr\,d\theta = \int_0^{2\pi} \frac{2}{3}r^3\Big|_0^{\sqrt{2}} \sin\theta\,d\theta = -\frac{4\sqrt{2}}{3}\cos\theta\Big|_0^{2\pi}$$

$$= \boxed{0}$$

15.6 **12.** Integrate $G(x, y, z) = xyz$ over the surface of the rectangular solid bounded by the planes $x = \pm a$, $y = \pm b$, and $z = \pm c$.

Sol'n:



When $x = a$. $G(a,y,z) = ayz$

and $\displaystyle\iint_{x=a} G(a,y,z)\,d\sigma = \int_{-c}^{c}\int_{-b}^{b} ayz\,dy\,dz$

$\displaystyle = \int_{-c}^{c} \frac{1}{2}a\cancel{y^2}\Big|_{-b}^{\cancel{b}\,0} z\,dz = \int_{-c}^{c} 0\,dz = 0.$

Similarly for $\displaystyle\iint_{x=-a} G(-a,y,z)\,d\sigma = \int_{-c}^{c}\int_{-b}^{b} -ayz\,dy\,dz$

$\displaystyle = \int_{-c}^{c} -\frac{1}{2}a\cancel{y^2}\Big|_{-b}^{\cancel{b}\,0} z\,dz = 0.$

We see that similarly, $\displaystyle\iint_{y=\pm b} G(x,\pm b,z)\,d\sigma = \iint_{z=\pm c} G(x,y,\pm c)\,d\sigma = 0.$

So $\displaystyle\iint_{\substack{\text{rectangular}\\ \text{solid}}} G(x,y,z)\,d\sigma = \boxed{0}$

15.6 **16.** Integrate $G(x, y, z) = x$ over the surface given by

$$z = x^2 + y \quad \text{for} \quad 0 \le x \le 1, \quad -1 \le y \le 1.$$

Sol'n: Surface is given as a graph $(x, y, f(x,y))$ where $f(x,y) = x^2 + y$.

Then $\vec{r}(x,y) = x\hat{i} + y\hat{j} + (x^2+y)\hat{k}$.

$\vec{r}_x = \hat{i} + 0\hat{j} + 2x\hat{k}$.

$\vec{r}_y = 0\hat{i} + \hat{j} + \hat{k}$.

So $\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2x \\ 0 & 1 & 1 \end{vmatrix} = -2x\hat{i} + \hat{j} + \hat{k}$

and $|\vec{r}_x \times \vec{r}_y| = \sqrt{4x^2 + 1 + 1} = \sqrt{4x^2+2}$

So $\iint\limits_S G(x,y,z)\, d\sigma = \int_0^1 \int_{-1}^1 x\sqrt{4x^2+2}\, dy\, dx = \int_0^1 2x\sqrt{4x^2+2}\, dx$

Let $u = 4x^2+2$. Then $du = 8x\,dx$.

$0 \le x \le 1$, then $2 \le u \le 6$.

$$= \int_2^6 \frac{1}{4}\sqrt{u}\,du = \frac{1}{6}u^{3/2}$$

$$= \frac{1}{6}\left(6\sqrt{6} - 2\sqrt{2}\right)$$

$$= \boxed{\sqrt{6} - \frac{\sqrt{2}}{3}}$$