15.3 **Testing for Conservative Fields**

Which fields in Exercises 1–6 are conservative, and which are not?

6. $\mathbf{F} = (e^x \cos y)\mathbf{i} - (e^x \sin y)\mathbf{j} + z\mathbf{k}$

Sol'n: Since natural domain of F is $\mathbb{R}^3$, it is open and simply connected. So it suffices to check that the PDEs in Cor 9 of lecture notes are satisfied.

$$\begin{cases} M = e^x \cos y \\ N = -e^x \sin y \\ L = z \end{cases}$$

Then $\frac{\partial M}{\partial y} = -e^x \sin y$, $\frac{\partial N}{\partial x} = -e^x \sin y$

$\frac{\partial N}{\partial z} = 0 \quad \frac{\partial L}{\partial y} = 0.$

$\frac{\partial L}{\partial x} = 0 \quad \frac{\partial M}{\partial z} = 0.$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, $\frac{\partial N}{\partial z} = \frac{\partial L}{\partial y}$, $\frac{\partial L}{\partial x} = \frac{\partial M}{\partial z}$, we verify that F is conservative

15.3

Finding Potential Functions

In Exercises 7–12, find a potential function f for the field $\mathbf{F}$.

7. $\mathbf{F} = 2x\mathbf{i} + 3y\mathbf{j} + 4z\mathbf{k}$

8. $\mathbf{F} = (y + z)\mathbf{i} + (x + z)\mathbf{j} + (x + y)\mathbf{k}$

9. $\mathbf{F} = e^{y+2z}(\mathbf{i} + x\mathbf{j} + 2x\mathbf{k})$

10. $\mathbf{F} = (y \sin z)\mathbf{i} + (x \sin z)\mathbf{j} + (xy \cos z)\mathbf{k}$

Soln: Need to find f s.t. $\nabla f = F$, i.e.

$$\frac{\partial f}{\partial x} = y\sin z, \quad \frac{\partial f}{\partial y} = x\sin z, \quad \frac{\partial f}{\partial z} = xy\cos z.$$

$$f = \int y\sin z \, dx = xy\sin z + g(y,z) + C$$

and $$\frac{\partial f}{\partial y} = x\sin z + \frac{\partial g}{\partial y} = x\sin z \Rightarrow \frac{\partial g}{\partial y} = 0.$$

So $f = xy\sin z + h(z)$.

and $$\frac{\partial f}{\partial h} = xy\cos z + \frac{\partial h}{\partial z} = xy\cos z \Rightarrow \frac{\partial h}{\partial z} = 0.$$

So $f(x,y,z) = xy\sin z + C$ for constant $C \in \mathbb{R}$.

15.3 12. $\mathbf{F} = \dfrac{y}{1+x^2y^2}\mathbf{i} + \left(\dfrac{x}{1+x^2y^2} + \dfrac{z}{\sqrt{1-y^2z^2}}\right)\mathbf{j} + \left(\dfrac{y}{\sqrt{1-y^2z^2}} + \dfrac{1}{z}\right)\mathbf{k}$

soln: $\dfrac{\partial f}{\partial x} = \dfrac{y}{1+x^2y^2}$, $\dfrac{\partial f}{\partial y} = \dfrac{x}{1+x^2y^2} + \dfrac{z}{\sqrt{1-y^2z^2}}$, $\dfrac{\partial f}{\partial z} = \dfrac{y}{\sqrt{1-y^2z^2}} + \dfrac{1}{z}$.

$f = \int \dfrac{y}{1+x^2y^2}\,dx = \arctan(xy) + g(y,z) + C_1$

$\dfrac{\partial f}{\partial y} = \dfrac{x}{1+x^2y^2} + \dfrac{z}{\sqrt{1-y^2z^2}} = \dfrac{x}{1+x^2y^2} + \dfrac{\partial g}{\partial y}$. $\Rightarrow \dfrac{\partial g}{\partial y} = \dfrac{z}{\sqrt{1-y^2z^2}}$

$\Rightarrow g = \int \dfrac{z}{\sqrt{1-y^2z^2}}\,dy = \arcsin(yz) + h(z) + C_2$.

$\dfrac{\partial f}{\partial z} = \dfrac{y}{\sqrt{1-y^2z^2}} + \dfrac{1}{z} \Rightarrow \dfrac{\partial h}{\partial z} = \dfrac{1}{z} \Rightarrow h = \int \dfrac{1}{z}\,dz = \ln|z| + C_3$

So $f(x,y,z) = \arctan(xy) + \arcsin(yz) + \ln|z| + C$ for constant $C \in \mathbb{R}$

15.3 **Exact Differential Forms**

In Exercises 13–17, show that the differential forms in the integrals are exact. Then evaluate the integrals.

16. $\displaystyle\int_{(0,0,0)}^{(3,3,1)} 2x\,dx - y^2\,dy - \frac{4}{1+z^2}\,dz$

Sol'n: Domain is open simply connected.

$M = 2x$, $N = -y^2$, $L = -\dfrac{4}{1+z^2}$.

Check $\dfrac{\partial M}{\partial z} = 0 = \dfrac{\partial L}{\partial x}$, $\dfrac{\partial L}{\partial y} = 0 = \dfrac{\partial N}{\partial z}$, $\dfrac{\partial N}{\partial x} = 0 = \dfrac{\partial M}{\partial y}$.

So $2x\,dx - y^2\,dy - \dfrac{4}{1+z^2}\,dz$ is exact.

So $\displaystyle\int_{(0,0,0)}^{(3,3,1)} 2x\,dx - y^2\,dy - \frac{4}{1+z^2}\,dz = f(3,3,1) - f(0,0,0)$ for some function f

s.t. $df = 2x\,dx - y^2\,dy - \dfrac{4}{1+z^2}\,dz$.

$$\Leftrightarrow \frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = -y^2, \quad \frac{\partial f}{\partial z} = \frac{-4}{1+z^2}$$

$$f = \int 2x\,dx = x^2 + g(y,z) + C_1$$

$$\Rightarrow \frac{\partial g}{\partial y} = -y^2 \Rightarrow g = \int -y^2\,dy = -\frac{1}{3}y^3 + h(z) + C_2.$$

$$\Rightarrow \frac{\partial h}{\partial z} = \frac{-4}{1+z^2} \Rightarrow h = \int \frac{-4}{1+z^2}\,dz = -4\arctan z + C_3.$$

So $f(x,y,z) = x^2 - \frac{1}{3}y^3 - 4\arctan z + C.$, $C \in \mathbb{R}$.

So $\displaystyle\int_{(0,0,0)}^{(3,3,1)} 2x\,dx - y^2\,dy - \frac{4}{1+z^2}\,dz = f(3,3,1) - f(0,0,0)$

$$= 3^2 - \frac{1}{3}3^3 - 4\arctan(1)$$

$$= \boxed{-\pi}$$

15.3

Finding Potential Functions to Evaluate Line Integrals

Although they are not defined on all of space R^3, the fields associated with Exercises 18–22 are conservative. Find a potential function for each field, and evaluate the integrals as in Example 6.

18. $\int_{(0,2,1)}^{(1,\pi/2,2)} 2\cos y\,dx + \left(\frac{1}{y} - 2x\sin y\right)dy + \frac{1}{z}\,dz$

Sol'n: Since field is conservative

$$\frac{\partial f}{\partial x} = 2\cos y \Rightarrow f = \int 2\cos y\,dx = 2x\cos y + g(y,z) + C_1$$

$$\frac{\partial f}{\partial y} = -2x\sin y + \frac{\partial g}{\partial y} \Rightarrow \frac{\partial g}{\partial y} = \frac{1}{y} \Rightarrow g = \ln|y| + h(z) + C_2.$$

$$\frac{\partial f}{\partial z} = \frac{\partial h}{\partial z} = \frac{1}{z} \Rightarrow h = \ln|z| + C_3.$$

So $f(x,y,z) = 2x\cos y + \ln|y| + \ln|z| + C.$

$$\text{Integral} = f(1,\tfrac{\pi}{2},2) - f(0,2,1) = 2\cdot\cancel{1\cdot\cos\tfrac{\pi}{2}}^{0} + \ln\left|\tfrac{\pi}{2}\right| + \cancel{\ln|2|} - \cancel{2\cdot 0\cdot\cos 2}^{0} - \cancel{\ln|2|} - \cancel{\ln|1|}^{0} = \boxed{\ln\tfrac{\pi}{2}}$$

15.3 **20.** $\int_{(1,2,1)}^{(2,1,1)} (2x\ln y - yz)\,dx + \left(\frac{x^2}{y} - xz\right)dy - xy\,dz$

Sol'n: $f = \int (2x\ln y - yz)\,dx = x^2\ln y - xyz + g(y,z) + C_1$

$\frac{\partial f}{\partial y} = \frac{x^2}{y} - xz + \frac{\partial g}{\partial y} = \frac{x^2}{y} - xz \Rightarrow \frac{\partial g}{\partial y} = 0.$

So $f = x^2\ln y - xyz + h(z) + C_1.$

$\frac{\partial f}{\partial z} = -xy + \frac{\partial h}{\partial z} = -xy \Rightarrow \frac{\partial h}{\partial z} = 0.$

$\Rightarrow f(x,y,z) = x^2\ln y - xyz + C.$

Integral $= f(2,1,1) - f(1,2,1) = 2^2 \cancel{\ln(1)}^{0} - \cancel{2\cdot 1}\cdot 1 - 1^2\ln(2) + \cancel{1\cdot 2}\cdot 1 = \boxed{-\ln 2}$

15.3 **22.** $\displaystyle\int_{(-1,-1,-1)}^{(2,2,2)} \frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2+y^2+z^2}$

Sol'n: $\displaystyle\frac{\partial f}{\partial x} = \frac{2x}{x^2+y^2+z^2}. \Rightarrow f = \int \frac{2x}{x^2+y^2+z^2}\,dx = \ln(x^2+y^2+z^2) + g(y,z) + C_1$

$$\frac{\partial f}{\partial x} = \frac{2x}{x^2+y^2+z^2} + \frac{\partial g}{\partial y} = \frac{2x}{x^2+y^2+z^2} \Rightarrow \frac{\partial g}{\partial y} = 0.$$

So $f = \ln(x^2+y^2+z^2) + h(z) + C_1$

$$\frac{\partial f}{\partial z} = \frac{2z}{x^2+y^2+z^2} + \frac{\partial h}{\partial z} = \frac{2z}{x^2+y^2+z^2} \Rightarrow \frac{\partial h}{\partial z} = 0$$

So $f(x,y,z) = \ln(x^2+y^2+z^2) + C.$

Integral $= f(2,2,2) - f(-1,-1,-1) = \ln(2^2+2^2+2^2) - \ln((-1)^2+(-1)^2+(-1)^2)$

$$= \ln 12 - \ln 3 = \boxed{\ln 4}$$

15.3 **Independence of path** Show that the values of the integrals in Exercises 25 and 26 do not depend on the path taken from A to B.

25. $\int_A^B z^2\,dx + 2y\,dy + 2xz\,dz$

26. $\int_A^B \frac{x\,dx + y\,dy + z\,dz}{\sqrt{x^2+y^2+z^2}}$

Sol'n: Suffices to show $F = \frac{x}{\sqrt{x^2+y^2+z^2}}\hat{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\hat{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\hat{k}$ is conservative.

$$=: M\hat{i} + N\hat{j} + L\hat{k}$$

$$\frac{\partial M}{\partial y} = -\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}\cdot 2xy = \frac{xy}{(x^2+y^2+z^2)^{3/2}} = \frac{\partial N}{\partial x}.$$

$$\frac{\partial N}{\partial z} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\cdot 2yz = \frac{yz}{(x^2+y^2+z^2)^{3/2}} = \frac{\partial L}{\partial y}.$$

$$\frac{\partial L}{\partial x} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\cdot 2xz = \frac{xz}{(x^2+y^2+z^2)^{3/2}} = \frac{\partial M}{\partial x}.$$

Since F is conservative, integral is path-independent.

15.3 In Exercises 27 and 28, find a potential function for $\mathbf{F}$.

27. $\mathbf{F} = \dfrac{2x}{y}\mathbf{i} + \left(\dfrac{1-x^2}{y^2}\right)\mathbf{j}, \qquad \{(x, y): y > 0\}$

28. $\mathbf{F} = (e^x \ln y)\mathbf{i} + \left(\dfrac{e^x}{y} + \sin z\right)\mathbf{j} + (y \cos z)\mathbf{k}$

Sol'n: $\dfrac{\partial f}{\partial x} = e^x \ln y \Rightarrow f = \int e^x \ln y \, dx = e^x \ln y + g(y,z) + C_1$

$\dfrac{\partial f}{\partial y} = \dfrac{e^x}{y} + \dfrac{\partial g}{\partial y} = \dfrac{e^x}{y} + \sin z. \Rightarrow \dfrac{\partial g}{\partial y} = \sin z$

$\Rightarrow g = \int \sin z \, dz = y \sin z + h(z) + C_2.$

$\dfrac{\partial f}{\partial z} = y\cos z + \dfrac{\partial h}{\partial z} = y \cos z \Rightarrow \dfrac{\partial h}{\partial z} = 0.$

So $f(x,y,z) = e^x \ln(y) + y \sin z + C$ where $\vec{\nabla} f = \vec{F}$.

153

38. Gravitational field

a. Find a potential function for the gravitational field

$$\mathbf{F} = -GmM\frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}}$$

(G, m, and M are constants).

b. Let P_1 and P_2 be points at distances s_1 and s_2 from the origin. Show that the work done by the gravitational field in part (a) in moving a particle from P_1 to P_2 is

$$GmM\left(\frac{1}{s_2} - \frac{1}{s_1}\right).$$

Sol'n: a) $\dfrac{\partial f}{\partial x} = \dfrac{-GmMx}{(x^2+y^2+z^2)^{3/2}}$.

$$\Rightarrow f = -GmM\int \frac{x}{(x^2+y^2+z^2)^{3/2}}\,dx = \frac{GmM}{\sqrt{x^2+y^2+z^2}} + g(y,z) + C_1.$$

$$\frac{\partial f}{\partial y} = \frac{-GmMy}{(x^2+y^2+z^2)^{3/2}} + \frac{\partial g}{\partial y} = \frac{-GmMy}{(x^2+y^2+z^2)^{3/2}} \Rightarrow \frac{\partial g}{\partial y} = 0.$$

So $f = \dfrac{GmM}{\sqrt{x^2+y^2+z^2}} + h(z) + c_1.$

$$\frac{\partial f}{\partial z} = \frac{-GmMz}{\sqrt{x^2+y^2+z^2}} + \frac{\partial h}{\partial z} = \frac{-GmMz}{\sqrt{x^2+y^2+z^2}} \Rightarrow \frac{\partial h}{\partial z} = 0.$$

So $f = \dfrac{GmM}{\sqrt{x^2+y^2+z^2}} + C.$

b) at P_1, $s_1 = \sqrt{x_1^2+y_1^2+z_1^2}$, P_2, $s_2 = \sqrt{x_2^2+y_2^2+z_2^2}$.

So work done $= \displaystyle\int_{P_1}^{P_2} F\,ds = f(s_2) - f(s_1) = GmM\left(\frac{1}{s_2} - \frac{1}{s_1}\right).$

//.