15.7

## Using Stokes' Theorem to Find Line Integrals

In Exercises 7–12, use the surface integral in Stokes' Theorem to calculate the circulation of the field **F** around the curve $C$ in the indicated direction.

7. $\mathbf{F} = x^2\mathbf{i} + 2x\mathbf{j} + z^2\mathbf{k}$

   $C$: The ellipse $4x^2 + y^2 = 4$ in the $xy$-plane, counterclockwise when viewed from above

8. $\mathbf{F} = 2y\mathbf{i} + 3x\mathbf{j} - z^2\mathbf{k}$

   $C$: The circle $x^2 + y^2 = 9$ in the $xy$-plane, counterclockwise when viewed from above

Sol'n: Circulation $= \oint_C \vec{F} \cdot d\vec{r} \underset{\text{Stoke's Thm}}{=} \iint_S (\vec{\nabla} \times \vec{F}) \cdot \vec{n}\, d\sigma$.

where $C$ = circle $x^2+y^2=9$. Then $C$ is boundary of $S: \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leq 9\}$.

Since we are in $xy$ plane, $\vec{n} = \hat{k}$,

$d\sigma = dxdy$.

$\vec{F} = 2y\hat{i} + 3x\hat{j} - z^2\hat{k}$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y & 3x & -z^2 \end{vmatrix}$$

$$= \left(\cancelto{0}{\frac{\partial}{\partial y}(-z^2)} - \cancelto{0}{\frac{\partial}{\partial z}(3x)}\right)\hat{i} - \left(\cancelto{0}{\frac{\partial}{\partial x}(-z^2)} - \cancelto{0}{\frac{\partial}{\partial z}(2y)}\right)\hat{j} + \left(\frac{\partial}{\partial x}(3x) - \frac{\partial}{\partial y}(2y)\right)\hat{k}$$

$$= (3-2)\hat{k} = \hat{k}.$$

So circulation $= \iint_S (\vec{\nabla}\times\vec{F})\cdot\vec{n}\,d\sigma = \iint_{x^2+y^2\le 9} \hat{k}\cdot\hat{k}\,dx\,dy = \iint_{x^2+y^2\le 9} dx\,dy = \boxed{9\pi}$

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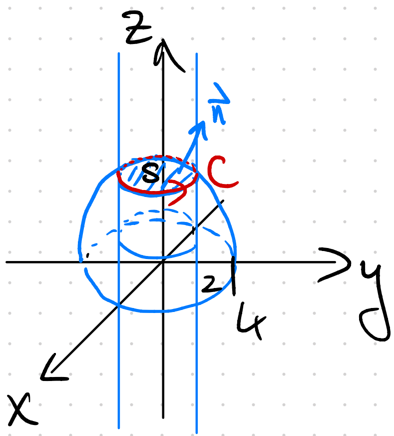
12. $\mathbf{F} = x^2y^3\mathbf{i} + \mathbf{j} + z\mathbf{k}$

$C$: The intersection of the cylinder $x^2 + y^2 = 4$ and the hemisphere $x^2 + y^2 + z^2 = 16$, $z \geq 0$, counterclockwise when viewed from above

Sol'n: Method 1: Parameterization: Circulation $= \oint_C \vec{F} \cdot d\vec{r} = \iint_S (\vec{\nabla} \times \vec{F}) \cdot \vec{n}\, d\sigma$.




At the intersection $4 + z^2 = 16 \Rightarrow z^2 = 12$

$\Rightarrow z = 2\sqrt{3}$ $(z \geq 0)$.

Taking polar coordinates $x = r\cos\theta$ $\quad 0 \leq \theta \leq 2\pi$

$y = r\sin\theta$ $\quad 0 \leq r \leq 2$

Then on $S$, we have

$x^2 + y^2 + z^2 = 16 \Rightarrow r^2 + z^2 = 16 \Rightarrow z = \sqrt{16 - r^2}$.

Then parameterization of $S$ given by

$\vec{r}(r, \theta) = r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j} + \sqrt{16 - r^2}\,\hat{k}$, $\quad 0 \leq \theta \leq 2\pi$, $0 \leq r \leq 2$.

$$\vec{r}_r = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath} - \frac{r}{\sqrt{16-r^2}}\hat{k}$$

$$\vec{r}_\theta = -r\sin\theta\,\hat{\imath} + r\cos\theta\,\hat{\jmath} + 0\hat{k}.$$

Then by $z \geq 0$, normal $\vec{n}$ is upward pointing, so

$$\vec{n} = + \frac{\vec{r}_r \times \vec{r}_\theta}{|\vec{r}_r \times \vec{r}_\theta|}.$$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \cos\theta & \sin\theta & \frac{-r}{\sqrt{16-r^2}} \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \frac{r^2\cos\theta}{\sqrt{16-r^2}}\hat{\imath} - \left(-\frac{-r}{\sqrt{16-r^2}} \cdot -r\sin\theta\right)\hat{\jmath}$$

$$+ \left(r\cos^2\theta + r\sin^2\theta\right)\hat{k}$$

$$= \frac{r^2\cos\theta}{\sqrt{16-r^2}}\hat{\imath} + \frac{r^2\sin\theta}{\sqrt{16-r^2}}\hat{\jmath} + r\hat{k}.$$

$$|\vec{r}_r \times \vec{r}_\theta| = \left(\frac{r^4\cos^2\theta}{16-r^2} + \frac{r^4\sin^2\theta}{16-r^2} + r^2\right)^{1/2} = \sqrt{\frac{r^4}{16-r^2}+r^2} = r\sqrt{\frac{r^2}{16-r^2}+1}$$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y^3 & 1 & z \end{vmatrix} = \left(\frac{\partial}{\partial x}(1) - \frac{\partial}{\partial y}(x^2y^3)\right)\hat{k} = -3x^2y^2\hat{k}$$

So $(\vec{\nabla}\times\vec{F})(\vec{r}(r,\theta)) = -3r^2\cos^2\theta\, r^2\sin^2\theta\,\hat{k} = -3r^4\cos^2\theta\sin^2\theta\,\hat{k}$.

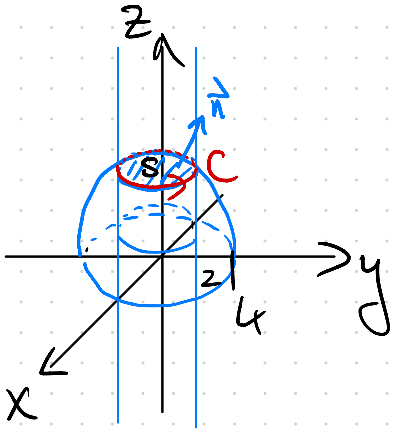
Then $\displaystyle\iint_S (\vec{\nabla}\times\vec{F})\cdot\vec{n}\,d\sigma = \int_0^{2\pi}\int_0^2 (\vec{\nabla}\times\vec{F})(\vec{r}(r,\theta))\cdot\frac{\vec{r}_r\times\vec{r}_\theta}{\cancel{|\vec{r}_r\times\vec{r}_\theta|}}\cancel{|\vec{r}_r\times\vec{r}_\theta|}\,dr\,d\theta$

$$= \int_0^{2\pi}\int_0^2 -3r^5\cos^2\theta\sin^2\theta\,dr\,d\theta$$

$$= \int_0^{2\pi}\left(\frac{-3}{6}r^6\Big|_0^2\right)\cos^2\theta\sin^2\theta\,d\theta = \int_0^{2\pi} -32\cos^2\theta\sin^2\theta\,d\theta = -32\cdot\frac{\pi}{4} = \boxed{-8\pi}$$

Method 2: S as level surface

$$\text{Circulation} = \oint_C \vec F\cdot d\vec r = \iint_S (\vec\nabla\times\vec F)\cdot\vec n\, d\sigma.$$




$$\vec\nabla\times\vec F = \begin{vmatrix} \hat i & \hat j & \hat k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y^3 & 1 & z \end{vmatrix}$$

$$= \left(\overset{0}{\cancel{\frac{\partial}{\partial x}(1)}} - \frac{\partial}{\partial y}(x^2y^3)\right)\hat k = -3x^2y^2\hat k.$$

$z \geq 0 \Rightarrow$ upward normal $\vec n$ (positive $\hat k$ component), so

$$\vec n = \frac{x\hat i + y\hat j + z\hat k}{\sqrt{x^2+y^2+z^2}} = \frac{x\hat i + y\hat j + z\hat k}{4}$$

$S$ is given as level surface of $g(x,y,z) = x^2+y^2+z^2 = 16 \quad (x^2+y^2 \leq 4)$

$\vec{\nabla} g = (2x, 2y, 2z)$. Since $z>0$, (at boundary $C$, $z^2 = 16-4 = 12 \Rightarrow z = 2\sqrt{3} > 0$)

$\frac{\partial g}{\partial z} = 2z \neq 0$.

Then $d\sigma = \frac{|\vec{\nabla} g|}{\left|\frac{\partial g}{\partial z}\right|} = \frac{\sqrt{4x^2+4y^2+4z^2}}{|2z|} dx\,dy = \frac{2\sqrt{4+12}}{|2z|} dx\,dy = \frac{4}{|z|} dx\,dy$.

$$\iint_S (\vec{\nabla} \times \vec{F}) \cdot \vec{n}\, d\sigma = \iint_{x^2+y^2 \leq 4} (-3x^2y^2\hat{k}) \cdot \frac{x\hat{i}+y\hat{j}+z\hat{k}}{4} \frac{4}{z} dx\,dy$$

$$= \iint_{x^2+y^2 \leq 4} \frac{-3x^2y^2\cancel{z}}{\cancel{4}} \cdot \frac{\cancel{4}}{\cancel{z}} dx\,dy = \iint_{x^2+y^2 \leq 4} -3x^2y^2\, dx\,dy$$

$$= \int_0^{2\pi}\int_0^{2} -3r^2\cos^2\theta\, r^2\sin^2\theta \; r\,dr\,d\theta = \int_0^{2\pi}\int_0^{2} -3r^5\cos^2\theta\sin^2\theta \, dr\,d\theta$$

$$= \int_0^{2\pi}\left(-\frac{3}{6}r^6\Big|_0^2\right)\cos^2\theta\sin^2\theta\, d\theta = \int_0^{2\pi} -32\cos^2\theta\sin^2\theta = \boxed{-8\pi}$$

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14. Let $\mathbf{n}$ be the unit normal in the direction away from the origin of the parabolic shell

$$S: \quad 4x^2 + y + z^2 = 4, \qquad y \geq 0,$$

and let

$$\mathbf{F} = \left(-z + \frac{1}{2+x}\right)\mathbf{i} + (\tan^{-1} y)\mathbf{j} + \left(x + \frac{1}{4+z}\right)\mathbf{k}.$$

Find the value of

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma.$$

Sol'n: By Stoke's Theorem, $\iint_S (\vec{\nabla}\times\vec{F})\cdot\vec{n}\,d\sigma = \oint_C \vec{F}\cdot d\vec{r}$

Boundary C of S given by $\begin{cases} y=0 \\ 4x^2+y+z^2=4 \end{cases} \Rightarrow 4x^2+z^2=4$

So parameterize C by $\begin{aligned} x &= \cos\theta \\ y &= 0 \\ z &= 2\sin\theta. \end{aligned} \quad 0 \leq \theta \leq 2\pi.$

i.e. by $\vec{r}(\theta) = (\cos\theta, 0, 2\sin\theta)$.

$$d\vec{r} = (-\sin\theta, 0, 2\cos\theta)$$

$$\vec{F}(\vec{r}(\theta)) = \left(-2\sin\theta + \frac{1}{2+\cos\theta},\ \tan^{-1}0,\ \cos\theta + \frac{1}{4+2\sin\theta}\right)$$

$$\vec{F}(\vec{r}(\theta))\cdot d\vec{r} = \left(-2\sin\theta + \frac{1}{2+\cos\theta},\ \tan^{-1}0,\ \cos\theta + \frac{1}{4+2\sin\theta}\right)\cdot(-\sin\theta, 0, 2\cos\theta)$$

$$= 2\sin^2\theta - \frac{\sin\theta}{2+\cos\theta} + 2\cos^2\theta + \frac{2\cos\theta}{4+2\sin\theta}$$

$$= 2 - \frac{\sin\theta}{2+\cos\theta} + \frac{2\cos\theta}{4+2\sin\theta}$$

So $$\iint_S (\vec{\nabla}\times\vec{F})\cdot\vec{n}\,d\sigma = \oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi}\left(2 - \frac{\sin\theta}{2+\cos\theta} + \frac{2\cos\theta}{4+2\sin\theta}\right)d\theta$$

$$= \int_0^{2\pi} 2\,d\theta + \int_0^{2\pi}\frac{-\sin\theta}{2+\cos\theta}\,d\theta + \int_0^{2\pi}\frac{2\cos\theta}{4+2\sin\theta}\,d\theta$$

$$= 2\theta\Big|_0^{2\pi} + \int_0^{2\pi} \frac{1}{2+\cos\theta}\, d(2+\cos\theta) + \int_0^{2\pi} \frac{1}{4+2\sin\theta}\, d(4+2\sin\theta)$$

$$= 4\pi + \ln|2+\cos\theta|\Big|_0^{2\pi} + \ln|4+2\sin\theta|\Big|_0^{2\pi}$$

$$= 4\pi + \cancel{\ln 3} - \cancel{\ln 3} + \cancel{\ln 4} - \cancel{\ln 4}$$

$$= \boxed{4\pi}$$

15.7 **28. Zero circulation** Let $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$. Show that the clockwise circulation of the field $\mathbf{F} = \nabla f$ around the circle $x^2 + y^2 = a^2$ in the $xy$-plane is zero

a. by taking $\mathbf{r} = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j}$, $0 \le t \le 2\pi$, and integrating $\mathbf{F} \cdot d\mathbf{r}$ over the circle.

b. by applying Stokes' Theorem.

Sol'n: $\vec{F} = \vec{\nabla} f = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2x, 2y, 2z)$

$= \frac{1}{(x^2+y^2+z^2)^{3/2}}(x, y, z)$

a) $\vec{r}(t) = (a\cos t, a\sin t, 0) \quad 0 \le t \le 2\pi.$

$d\vec{r} = (-a\sin t, a\cos t, 0).$

$\vec{F}(\vec{r}(t)) = \frac{1}{(a^2)^{3/2}}(a\cos t, a\sin t, 0)$

$\vec{F}(\vec{r}(t)) \cdot d\vec{r} = (a^{-2}\cos t, a^{-2}\sin t, 0) \cdot (-a\sin t, a\cos t, 0).$

$= -a\sin t\cos t + a\sin t\cos t = 0.$

So circulation $= \oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t))\cdot d\vec{r} = \int_0^{2\pi} 0 \, dt = 0.$

Note, could have also noted that $\vec{F} = \vec{\nabla} f$ is a conservative vector field, hence $\oint_C \vec{F}\cdot d\vec{r} = 0.$

b) Circle in xy plane and clockwise circulation, so $\vec{n} = -\hat{k}$.

$S = \{(x,y) : x^2+y^2 \leq a^2\}$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{(x^2+y^2+z^2)^{3/2}} & \frac{y}{(x^2+y^2+z^2)^{3/2}} & \frac{z}{(x^2+y^2+z^2)^{3/2}} \end{vmatrix} = 0.$$

So circulation $= \oint_C \vec{F} \cdot d\vec{r} \underset{\text{Stoke's Thm}}{=} \iint_{x^2+y^2 \leq a^2} 0 \cdot (-\hat{k}) \, dx\,dy = 0$. //.

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**34. Zero curl, yet the field is not conservative** Show that the curl of

$$\mathbf{F} = \frac{-y}{x^2 + y^2}\mathbf{i} + \frac{x}{x^2 + y^2}\mathbf{j} + z\mathbf{k}$$

is zero but that

$$\oint_C \mathbf{F} \cdot d\mathbf{r}$$

is not zero if $C$ is the circle $x^2 + y^2 = 1$ in the $xy$-plane. (Theorem 7 does not apply here because the domain of $\mathbf{F}$ is not simply connected. The field $\mathbf{F}$ is not defined along the $z$-axis, so there is no way to contract $C$ to a point without leaving the domain of $\mathbf{F}$.)

Sol'n:

$$\text{Curl} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & z \end{vmatrix} = \left(\cancel{\frac{\partial}{\partial y}(z)} - \cancel{\frac{\partial}{\partial z}\left(\frac{x}{x^2+y^2}\right)}\right)\hat{i} - \left(\cancel{\frac{\partial}{\partial x}(z)} - \cancel{\frac{\partial}{\partial z}\left(\frac{-y}{x^2+y^2}\right)}\right)\hat{j}$$

$$+ \left(\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)\right)\hat{k}$$

$$= \frac{(x^2+y^2)1 - x(2x)}{(x^2+y^2)^2} - \frac{(x^2+y^2)(-1) - (-y)(2y)}{(x^2+y^2)^2}\hat{k}$$

$$= \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} - \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2}\hat{k}$$

$$= \frac{-x^2+y^2}{(x^2+y^2)^2} - \frac{-x^2+y^2}{(x^2+y^2)^2}\hat{k} = 0$$

But

$$\vec{r}(\theta) = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath} + 0\hat{k}. \qquad 0 \le \theta \le 2\pi$$

$$d\vec{r} = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath} + 0\hat{k}$$

$$\vec{F}(\vec{r}(\theta)) = \frac{-\sin\theta}{1}\hat{\imath} + \frac{\cos\theta}{1}\hat{\jmath} + 0\hat{k}$$

Then $\vec{F}(\vec{r}(\theta)) \cdot d\vec{r} = (-\sin\theta, \cos\theta, 0) \cdot (-\sin\theta, \cos\theta, 0)$

$$= \sin^2\theta + \cos^2\theta = 1.$$

So $\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(\theta)) \cdot d\vec{r} = \int_0^{2\pi} d\theta = 2\pi \neq 0.$

15.8

## Calculating Divergence

In Exercises 1–8, find the divergence of the field.

1. $\mathbf{F} = (x - y + z)\mathbf{i} + (2x + y - z)\mathbf{j} + (3x + 2y - 2z)\mathbf{k}$
2. $\mathbf{F} = (x \ln y)\mathbf{i} + (y \ln z)\mathbf{j} + (z \ln x)\mathbf{k}$
3. $\mathbf{F} = ye^{xyz}\mathbf{i} + ze^{xyz}\mathbf{j} + xe^{xyz}\mathbf{k}$
4. $\mathbf{F} = \sin(xy)\mathbf{i} + \cos(yz)\mathbf{j} + \tan(xz)\mathbf{k}$

Sol'n: $\operatorname{div} F = \vec{\nabla} \cdot \vec{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left(\sin(xy), \cos(yz), \tan(xz)\right)$

$= y\cos(xy) - z\sin(yz) + x\sec^2(xz)$

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## Calculating Flux Using the Divergence Theorem

In Exercises 9–20, use the Divergence Theorem to find the outward flux of **F** across the boundary of the region $D$.

**12. Ball** $\mathbf{F} = x^2\mathbf{i} + xz\mathbf{j} + 3z\mathbf{k}$

$D$: The ball $x^2 + y^2 + z^2 \le 4$

Sol'n: By divergence theorem, $\text{Flux} = \iint_S \vec{F}\cdot\vec{n}\,d\sigma = \iiint_D \vec{\nabla}\cdot\vec{F}\,dV$

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(3z) = 2x+3.$$

So
$$\text{Flux} = \iiint_{x^2+y^2+z^2\le 4} 2x+3\,dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{2}(2\rho\sin\phi\cos\theta+3)(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{2}(2\rho^3\sin^2\phi\cos\theta + 3\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi}\left(\left(\tfrac{1}{2}\rho^4\right)\Big|_0^2\sin^2\phi\cos\theta + (\rho^3)\Big|_0^2\sin\phi\right)d\phi\,d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} \left( 8 \sin^2\phi \cos\theta + 8 \sin\phi \right) d\phi \, d\theta$$

$$= \int_0^{2\pi} \left( 4\pi \cos\theta + 16 \right) d\theta = \boxed{32\pi}$$

15.8 **18. Thick sphere** $\mathbf{F} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})/\sqrt{x^2+y^2+z^2}$

*D*: The region $1 \le x^2 + y^2 + z^2 \le 4$

Sol'n: $$\vec{\nabla}\cdot\vec{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot \frac{1}{(x^2+y^2+z^2)^{1/2}}(x,y,z)$$

$$= \frac{1}{(x^2+y^2+z^2)^{3/2}}\left(y^2+z^2+x^2+z^2+x^2+y^2\right) = \frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{3/2}} = \frac{2}{\sqrt{x^2+y^2+z^2}}$$

$= \frac{2}{\rho}$ in spherical coords, $1 \le \rho \le 2$.

So flux $$= \iiint_D \vec{\nabla}\cdot\vec{F}\,dV = \int_0^{2\pi}\int_0^{\pi}\int_1^2 \frac{2}{\cancel{\rho}}\rho^{\cancel{2}}\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_1^2 2\rho\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi} \rho^2\Big|_1^2 \sin\phi\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} 3\sin\phi\,d\phi\,d\theta = \int_0^{2\pi} 3(-\cos\phi)\Big|_0^{\pi}\,d\theta = \int_0^{2\pi} 6\,d\theta = \boxed{12\pi}$$

15.8 **30.** Let $\mathbf{F}_1$ and $\mathbf{F}_2$ be differentiable vector fields, and let $a$ and $b$ be arbitrary real constants. Verify the following identities.

**a.** $\nabla \cdot (a\mathbf{F}_1 + b\mathbf{F}_2) = a\nabla \cdot \mathbf{F}_1 + b\nabla \cdot \mathbf{F}_2$

**b.** $\nabla \times (a\mathbf{F}_1 + b\mathbf{F}_2) = a\nabla \times \mathbf{F}_1 + b\nabla \times \mathbf{F}_2$

**c.** $\nabla \cdot (\mathbf{F}_1 \times \mathbf{F}_2) = \mathbf{F}_2 \cdot \nabla \times \mathbf{F}_1 - \mathbf{F}_1 \cdot \nabla \times \mathbf{F}_2$

Sol'n: a) $\vec{F}_1 = M_1\hat{i} + N_1\hat{j} + L_1\hat{k}$, $\vec{F}_2 = M_2\hat{i} + N_2\hat{j} + L_2\hat{k}$

$$a\vec{F}_1 + b\vec{F}_2 = (aM_1 + bM_2, aN_1 + bN_2, aL_1 + bL_2).$$

Then $\vec{\nabla}\cdot(a\vec{F}_1 + b\vec{F}_2) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot(aM_1 + bM_2, aN_1 + bN_2, aL_1 + bL_2)$

$$= a\frac{\partial M_1}{\partial x} + b\frac{\partial M_2}{\partial x} + a\frac{\partial N_1}{\partial y} + b\frac{\partial N_2}{\partial y} + a\frac{\partial L_1}{\partial z} + b\frac{\partial L_2}{\partial z}$$

$$= a\frac{\partial M_1}{\partial x} + a\frac{\partial N_1}{\partial y} + a\frac{\partial L_1}{\partial z} + b\frac{\partial M_2}{\partial x} + b\frac{\partial N_2}{\partial y} + b\frac{\partial L_2}{\partial z}$$

$$= a\left(\frac{\partial M_1}{\partial x} + \frac{\partial N_1}{\partial y} + \frac{\partial L_1}{\partial z}\right) + b\left(\frac{\partial M_2}{\partial x} + \frac{\partial N_2}{\partial y} + \frac{\partial L_2}{\partial z}\right)$$

$$= a\vec{\nabla}\cdot\vec{F}_1 + b\vec{\nabla}\cdot\vec{F}_2 . \quad \checkmark$$

b) $\vec{\nabla}\times(a\vec{F}_1 + b\vec{F}_2)$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ aM_1+bM_2 & aN_1+bN_2 & aL_1+bL_2 \end{vmatrix} = \left(\frac{\partial}{\partial y}(aL_1+bL_2) - \frac{\partial}{\partial z}(aN_1+bN_2)\right)\hat{i}$$

$$-\left(\frac{\partial}{\partial x}(aL_1+bL_2) - \frac{\partial}{\partial z}(aM_1+bM_2)\right)\hat{j}$$

$$+\left(\frac{\partial}{\partial x}(aN_1+bN_2) - \frac{\partial}{\partial y}(aM_1+bM_2)\right)\hat{k}$$

$$= \left(a\frac{\partial L_1}{\partial y} + b\frac{\partial L_2}{\partial y} - a\frac{\partial N_1}{\partial z} - b\frac{\partial N_2}{\partial z}\right)\hat{i} - \left(a\frac{\partial L_1}{\partial x} + b\frac{\partial L_2}{\partial x} - a\frac{\partial M_1}{\partial z} - b\frac{\partial M_2}{\partial z}\right)\hat{j}$$

$$+\left(a\frac{\partial N_1}{\partial x} + b\frac{\partial N_2}{\partial x} - a\frac{\partial M_1}{\partial y} - b\frac{\partial M_2}{\partial y}\right)\hat{k}$$

$$= a\left(\frac{\partial L_1}{\partial y} - \frac{\partial N_1}{\partial z}\right)\hat{\imath} - a\left(\frac{\partial L_1}{\partial x} - \frac{\partial M_1}{\partial z}\right)\hat{\jmath} + a\left(\frac{\partial N_1}{\partial x} - \frac{\partial M_1}{\partial y}\right)\hat{k}$$

$$+ b\left(\frac{\partial L_2}{\partial y} - \frac{\partial N_2}{\partial z}\right)\hat{\imath} - b\left(\frac{\partial L_2}{\partial x} - \frac{\partial M_2}{\partial z}\right)\hat{\jmath} + b\left(\frac{\partial N_2}{\partial x} - \frac{\partial M_2}{\partial y}\right)\hat{k}$$

$$= a\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M_1 & N_1 & L_1 \end{vmatrix} + b\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M_2 & N_2 & L_2 \end{vmatrix}$$

$$= a\vec{\nabla}\times\vec{F}_1 + b\vec{\nabla}\times\vec{F}_2.$$

$\vec{\nabla}\cdot\left(\vec{F}_1\times\vec{F}_2\right)$: $\quad \vec{F}_1\times\vec{F}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ M_1 & N_1 & L_1 \\ M_2 & N_2 & L_2 \end{vmatrix}$

$$= (N_1 L_2 - L_1 N_2)\hat{i} - (M_1 L_2 - L_1 M_2)\hat{j} + (M_1 N_2 - N_1 M_2)\hat{k}$$

So $\vec{\nabla} \cdot (\vec{F}_1 \times \vec{F}_2) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot \left(N_1 L_2 - L_1 N_2, -(M_1 L_2 - L_1 M_2), M_1 N_2 - N_1 M_2\right)$

$$= \frac{\partial N_1}{\partial x} L_2 + N_1 \frac{\partial L_2}{\partial x} - \frac{\partial L_1}{\partial x} N_2 - L_1 \frac{\partial N_2}{\partial x} - \left(\frac{\partial M_1}{\partial y} L_2 + M_1 \frac{\partial L_2}{\partial y} - \frac{\partial L_1}{\partial y} M_2 - L_1 \frac{\partial M_2}{\partial y}\right)$$

$$+ \frac{\partial M_1}{\partial z} N_2 + M_1 \frac{\partial N_2}{\partial z} - \frac{\partial N_1}{\partial z} M_2 - N_1 \frac{\partial M_2}{\partial z}$$

$$= M_2 \left(\frac{\partial L_1}{\partial y} - \frac{\partial N_1}{\partial z}\right) + N_2 \left(\frac{\partial M_1}{\partial z} - \frac{\partial L_1}{\partial x}\right) + L_2 \left(\frac{\partial N_1}{\partial x} - \frac{\partial M_1}{\partial y}\right)$$

$$- M_1 \left(\frac{\partial L_2}{\partial y} - \frac{\partial N_2}{\partial z}\right) - N_1 \left(\frac{\partial M_2}{\partial z} - \frac{\partial L_2}{\partial x}\right) - L_1 \left(\frac{\partial N_2}{\partial x} - \frac{\partial M_2}{\partial y}\right)$$

$$= \vec{F}_2 \cdot \left(\frac{\partial L_1}{\partial y} - \frac{\partial N_1}{\partial z}, \frac{\partial M_1}{\partial z} - \frac{\partial L_1}{\partial x}, \frac{\partial N_1}{\partial x} - \frac{\partial M_1}{\partial y}\right) - \vec{F}_1 \cdot \left(\frac{\partial L_2}{\partial y} - \frac{\partial N_2}{\partial z}, \frac{\partial M_2}{\partial z} - \frac{\partial L_2}{\partial x}, \frac{\partial N_2}{\partial x} - \frac{\partial M_2}{\partial y}\right)$$

$$= \vec{F}_2 \cdot \vec{\nabla} \times \vec{F}_1 - \vec{F}_1 \cdot \vec{\nabla} \times \vec{F}_2$$

15.8

**32. Harmonic functions** A function $f(x, y, z)$ is said to be *harmonic* in a region $D$ in space if it satisfies the Laplace equation

$$\nabla^2 f = \nabla \cdot \nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

throughout $D$.

**a.** Suppose that $f$ is harmonic throughout a bounded region $D$ enclosed by a smooth surface $S$ and that $\mathbf{n}$ is the chosen unit normal vector on $S$. Show that the integral over $S$ of $\nabla f \cdot \mathbf{n}$, the derivative of $f$ in the direction of $\mathbf{n}$, is zero.

**b.** Show that if $f$ is harmonic on $D$, then

$$\iint_S f \nabla f \cdot \mathbf{n}\, d\sigma = \iiint_D |\nabla f|^2\, dV.$$

Sol'n: a) By Divergence theorem,

$$\iint_S (\vec{\nabla} f \cdot \vec{n})\, d\sigma = \iiint_D \operatorname{div} \vec{\nabla} f\, dV = \iiint_D \overbrace{\vec{\nabla} \cdot \vec{\nabla} f}^{0}\, dV = 0$$

Since f is harmonic.

b) $$\iint_S f\vec{\nabla}f\cdot\vec{n}\,d\sigma \underset{\substack{\uparrow\\ \text{divergence}\\ \text{thm}}}{=} \iiint_D \vec{\nabla}\cdot\left(f\vec{\nabla}f\right)dV = \iiint_D\left(\vec{\nabla}f\cdot\vec{\nabla}f + f\,\cancelto{0}{\vec{\nabla}\cdot\vec{\nabla}f}\right)dV$$

$$= \iiint_D |\vec{\nabla}f|^2\,dV.$$