15.5

Surface Area of Parametrized Surfaces

In Exercises 17–26, use a parametrization to express the area of the surface as a double integral. Then evaluate the integral. (There are many correct ways to set up the integrals, so your integrals may not be the same as those in the back of the text. They should have the same values, however.)

17. Tilted plane inside cylinder The portion of the plane $y + 2z = 2$ inside the cylinder $x^2 + y^2 = 1$

18. Plane inside cylinder The portion of the plane $z = -x$ inside the cylinder $x^2 + y^2 = 4$

19. Cone frustum The portion of the cone $z = 2\sqrt{x^2 + y^2}$ between the planes $z = 2$ and $z = 6$

Using cylindrical coordinates

$x = r\cos\theta$, $0 \le \theta \le 2\pi$.

$y = r\sin\theta$

$z = 2\sqrt{x^2+y^2} = 2r$.

$2 \le z \le 6$

$\Updownarrow$

$2 \le 2r \le 6 \Leftrightarrow 1 \le r \le 3$.

So parameterization is given by

$\vec{r}(r,\theta) = r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j} + 2r\hat{k}$.

$$A = \int_0^{2\pi}\int_1^3 |\vec{r}_r \times \vec{r}_\theta|\, dr\, d\theta. \qquad |\vec{r}_r \times \vec{r}_\theta| = \left\| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 2 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} \right\|$$

$$= \left| -2r\cos\theta\,\hat{i} - 2r\sin\theta\,\hat{j} + (r\cos^2\theta + r\sin^2\theta)\,\hat{k} \right|$$

$$= \left| -2r\cos\theta\,\hat{i} - 2r\sin\theta\,\hat{j} + r\hat{k} \right|$$

$$= \left((4r^2\cos^2\theta) + (4r^2\sin^2\theta) + r^2 \right)^{1/2}$$

$$= \sqrt{5r^2} = r\sqrt{5}.$$

$$\text{so } A = \int_0^{2\pi}\int_1^3 r\sqrt{5}\, dr\, d\theta = 2\pi\sqrt{5}\,\frac{1}{2}r^2\Big|_1^3 = 9\sqrt{5}\pi - \sqrt{5}\pi = \boxed{8\sqrt{5}\pi}$$

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Planes Tangent to Parametrized Surfaces

The tangent plane at a point $P_0(f(u_0, v_0), g(u_0, v_0), h(u_0, v_0))$ on a parametrized surface $\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}$ is the plane through P_0 normal to the vector $\mathbf{r}_u(u_0, v_0) \times \mathbf{r}_v(u_0, v_0)$, the cross product of the tangent vectors $\mathbf{r}_u(u_0, v_0)$ and $\mathbf{r}_v(u_0, v_0)$ at P_0. In Exercises 27–30, find an equation for the plane tangent to the surface at P_0. Then find a Cartesian equation for the surface, and sketch the surface and tangent plane together.

27. Cone The cone $\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}$, $r \geq 0$, $0 \leq \theta \leq 2\pi$ at the point $P_0(\sqrt{2}, \sqrt{2}, 2)$ corresponding to $(r, \theta) = (2, \pi/4)$

28. Hemisphere The hemisphere surface $\mathbf{r}(\phi, \theta) = (4\sin\phi\cos\theta)\mathbf{i} + (4\sin\phi\sin\theta)\mathbf{j} + (4\cos\phi)\mathbf{k}$, $0 \leq \phi \leq \pi/2$, $0 \leq \theta \leq 2\pi$, at the point $P_0(\sqrt{2}, \sqrt{2}, 2\sqrt{3})$ corresponding to $(\phi, \theta) = (\pi/6, \pi/4)$

29. Circular cylinder The circular cylinder $\mathbf{r}(\theta, z) = (3\sin 2\theta)\mathbf{i} + (6\sin^2\theta)\mathbf{j} + z\mathbf{k}$, $0 \leq \theta \leq \pi$, at the point $P_0(3\sqrt{3}/2, 9/2, 0)$ corresponding to $(\theta, z) = (\pi/3, 0)$ (See Example 3.)

$$\vec{r}(\theta, z) = 3\sin 2\theta\,\hat{i} + 6\sin^2\theta\,\hat{j} + z\hat{k}, \quad 0 \leq \theta \leq \pi.$$

$$\vec{r}_\theta = 6\cos 2\theta\,\hat{i} + 12\sin\theta\cos\theta\,\hat{j} + 0\hat{k}.$$

$$\vec{r}_z = 0\hat{i} + 0\hat{j} + \hat{k}.$$

So $\vec{r}_\theta \times \vec{r}_z = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 6\cos 2\theta & 12\sin\theta\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = 12\sin\theta\cos\theta\,\hat{i} - 6\cos 2\theta\,\hat{j} + 0\hat{k}.$

at the point $(\theta, z) = (\frac{\pi}{3}, 0)$,

$$\vec{r}_\theta \times \vec{r}_z \Big|_{(\frac{\pi}{3},0)} = 3\sqrt{3}\,\hat{i} + 3\hat{j} + 0\hat{k}.$$

So tangent plane given by

$$3\sqrt{3}\left(x - \frac{3\sqrt{3}}{2}\right) + 3\left(y - \frac{9}{2}\right) + \cancel{0(z-0)} = 0$$

$$3\sqrt{3}x - \frac{27}{2} + 3y - \frac{27}{2} = 0.$$

$$\Leftrightarrow \quad 3\sqrt{3}x + 3y = 27$$

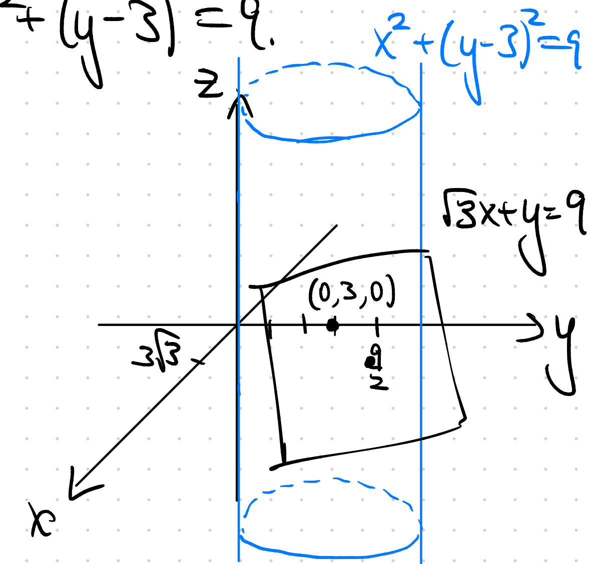
$$\Updownarrow$$

$$\sqrt{3}x + y = 9$$

Cartesian Equation:

by example 3, this is given by

$$x^2 + (y-3)^2 = 9.$$



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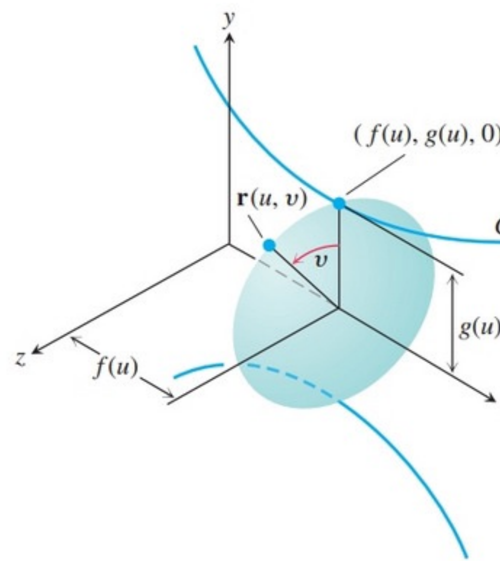
32. Parametrization of a surface of revolution Suppose that the parametrized curve C: $(f(u), g(u))$ is revolved about the x-axis, where $g(u) > 0$ for $a \le u \le b$.

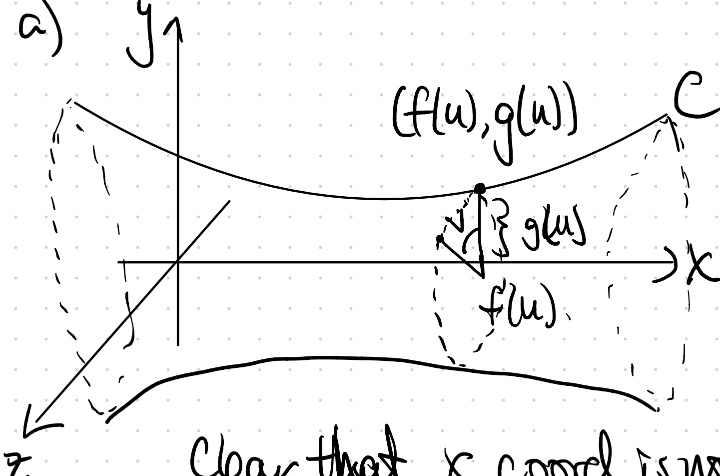
a. Show that

$$\mathbf{r}(u, v) = f(u)\mathbf{i} + (g(u)\cos v)\mathbf{j} + (g(u)\sin v)\mathbf{k}$$

is a parametrization of the resulting surface of revolution, where $0 \le v \le 2\pi$ is the angle from the xy-plane to the point $\mathbf{r}(u, v)$ on the surface. (See the accompanying figure.) Notice that $f(u)$ measures distance *along* the axis of revolution and $g(u)$ measures distance *from* the axis of revolution.

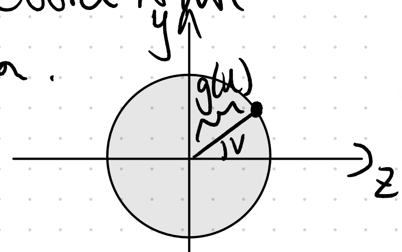


a)



Clear that x coord is not changed in the revolution.

Clear that in yz plane, point lies on circle



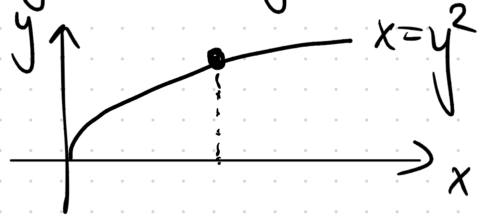
b. Find a parametrization for the surface obtained by revolving the curve $x = y^2$, $y \ge 0$, about the x-axis.

with radius $g(u)$. So $y = g(u)\cos v$, $z = g(u)\sin v$.
where v is the angle. $(0 \le v \le 2\pi)$.

So $\vec{r}(u,v) = f(u)\,\hat{i} + g(u)\cos v\,\hat{j} + g(u)\sin v\,\hat{k}$.

b) $x = y^2$, $y \ge 0$.

$\Leftrightarrow y = \sqrt{x}$



So curve is given by $(u, \sqrt{u})$, $u \ge 0$ $(y \ge 0 \Rightarrow \sqrt{u} \ge 0 \Rightarrow u \ge 0)$.
and surface of revolution given by

$$\vec{r}(u,v) = u\,\hat{i} + \sqrt{u}\cos v\,\hat{j} + \sqrt{u}\sin v\,\hat{k}$$

15.6 Integrate the given function over the given surface

7. Parabolic dome $H(x, y, z) = x^2\sqrt{5 - 4z}$, over the parabolic dome $z = 1 - x^2 - y^2$, $z \geq 0$

Surface param by: $x = r\cos\theta$ $\quad z = 1 - r^2 \quad (0 \leq r \leq 1)$.

$y = r\sin\theta$.

$z = z$.

So $\vec{r}(r,\theta) = r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j} + (1-r^2)\hat{k}$

$\vec{r}_r = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} - 2r\hat{k}$

$\vec{r}_\theta = -r\sin\theta\,\hat{i} + r\cos\theta\,\hat{j} + 0\hat{k}$.

$$|\vec{r}_r \times \vec{r}_\theta| = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & -2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \left|2r^2\cos\theta\,\hat{i} + 2r^2\sin\theta\,\hat{j} + (r\cos^2\theta + r\sin^2\theta)\hat{k}\right|$$

$$= \sqrt{4r^4\cos^2\theta + 4r^4\sin^2\theta + r^2} = \sqrt{4r^4 + r^2} = \sqrt{r^2(4r^2+1)} = r\sqrt{4r^2+1}$$

$$H(x,y,z) = r^2\cos^2\theta\sqrt{5-4(1-r^2)} = r^2\cos^2\theta\sqrt{5-4+4r^2} = r^2\cos^2\theta\sqrt{1+4r^2}$$

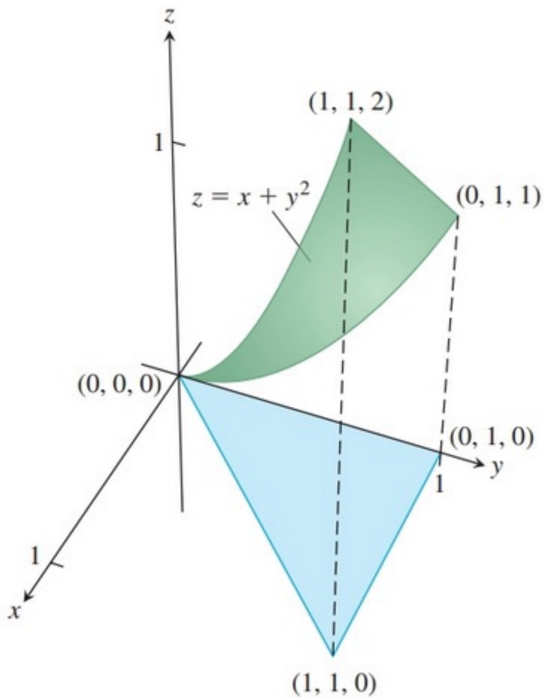
So $$\iint_S x^2\sqrt{5-4z}\,d\sigma = \int_0^{2\pi}\int_0^1 \left(r^2\cos^2\theta\sqrt{1+4r^2}\cdot r\sqrt{1+4r^2}\right)dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 r^3\cos^2\theta(1+4r^2)\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \left(r^3\cos^2\theta + 4r^5\cos^2\theta\right)dr\,d\theta$$

$$= \int_0^{2\pi}\left(\frac{1}{4}r^4\cos^2\theta + \frac{4}{6}r^6\cos^2\theta\right)\Big|_0^1 d\theta = \int_0^{2\pi}\left(\frac{1}{4}\cos^2\theta + \frac{4}{6}\cos^2\theta\right)d\theta$$

$$= \frac{11}{12}\int_0^{2\pi}\cos^2\theta\,d\theta = \boxed{\frac{11\pi}{12}}$$

15.6 **15.** Integrate $G(x, y, z) = z - x$ over the portion of the graph of $z = x + y^2$ above the triangle in the xy-plane having vertices $(0, 0, 0)$, $(1, 1, 0)$, and $(0, 1, 0)$. (See accompanying figure.)



Triangle: $0 \le y \le 1$.
$0 \le x \le y$.

$f(x,y) = x + y^2$.

$G(x,y,f(x,y)) = x + y^2 - x$
$= y^2$.

$\vec{\nabla} f = (1, 2y)$

$|\vec{\nabla} f|^2 = 1^2 + (2y)^2 = 1 + 4y^2$.

$$\iint_S G(x,y,z)\,d\sigma = \int_0^1 \int_0^y y^2 \sqrt{1 + 1 + 4y^2}\,dx\,dy = \int_0^1 x\Big|_0^y\, y^2 \sqrt{2 + 4y^2}\,dy$$

$$= \int_0^1 y^3 \sqrt{2+4y^2}\, dy = \boxed{\frac{1}{30}\left(\sqrt{2} + 6\sqrt{6}\right)}$$