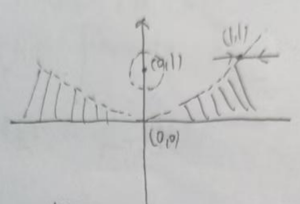
eg: $f:\mathbb{R}^2\to\mathbb{R}$ defined by $f(x,y)=\begin{cases}1, & \text{if } 0<y<x^2\\ 0, & \text{otherwise}\end{cases}$, we considre three points

(i) $\vec{a}=(0,1)$,

(ii) $\vec{a}=(1,1)$,

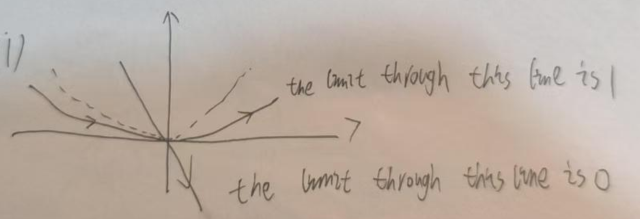
(iii) $\vec{a}=(0,0)$.



(i) For $\vec{a}=(0,1)$, the only possible result is 0.

(ii) For $\vec{a}=(1,1)$, it is easy to see the limit from the left is 0 and the limit from the right is 1

(iii)



Properties of limits

Assuming all limits on the right hand side exist, then the limit on the left hand side exists and the formula holds

(1) $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) \pm g(\vec{x}) = \lim_{\vec{x}\to\vec{a}} f(\vec{x}) \pm \lim_{\vec{x}\to\vec{a}} g(\vec{x})$.

(2) $\lim_{\vec{x}\to\vec{a}} kf(\vec{x}) = k\lim_{\vec{x}\to\vec{a}} f(\vec{x})$, where k is a constant.

(3) $\lim_{\vec{x}\to\vec{a}} f(\vec{x})g(\vec{x}) = \lim_{\vec{x}\to\vec{a}} f(\vec{x}) \lim_{\vec{x}\to\vec{a}} g(\vec{x})$,

(4) $\lim_{\vec{x}\to\vec{a}} \frac{f(\vec{x})}{g(\vec{x})} = \frac{\lim_{\vec{x}\to\vec{a}} f(\vec{x})}{\lim_{\vec{x}\to\vec{a}} g(\vec{x})}$ if $\lim_{\vec{x}\to\vec{a}} g(\vec{x}) \neq 0$.

(5) $\lim_{\vec{x}\to\vec{a}} [f(\vec{x})]^n = [\lim_{\vec{x}\to\vec{a}} f(\vec{x})]^n$, $n \geq 0$ (integer)

(6) $\lim_{\vec{x}\to\vec{a}} [f(\vec{x})]^{\frac{1}{n}} = [\lim_{\vec{x}\to\vec{a}} f(\vec{x})]^{\frac{1}{n}}$, $n \geq 0$ (integer)

Squeeze Theorem.

If $\lim_{\vec{x}\to\vec{a}} g(\vec{x}) = \lim_{\vec{x}\to\vec{a}} h(\vec{x}) = L$, and $g(\vec{x}) \le f(\vec{x}) \le h(\vec{x})$, then the limit $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = L$ holds.

We have the following special theorem,

If $\begin{cases} |f(\vec{x})| \le g(\vec{x}) \text{ near } \vec{a} \text{ and,} \\ \lim_{\vec{x}\to\vec{a}} g(\vec{x}) = 0, \end{cases}$

then $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = 0$

The application of the squeeze theorem.

$$\lim_{(x,y)\to(0,0)} x\cos\left(\frac{1}{x^2+y^2}\right)$$

In this step, it is very hard to decide whether the limit $(x,y)\to(0,0)$. We will learn some other limit criteria sometimes later. Here we can use the squeeze theorem.

$$\left|x\cos\left(\frac{1}{x^2+y^2}\right)\right| = |x| \cos\left(\frac{1}{x^2+y^2}\right) \le |x|$$

So $\lim_{(x,y)\to(0,0)} \left|x\cos\left(\frac{1}{x^2+y^2}\right)\right| = 0$.

Question, is the limit $\lim_{(x,y)\to(0,0)} \cos\left(\frac{1}{x^2+y^2}\right)$ exists?

eg: $\lim\limits_{(x,y)\to(1,0)} \left|\frac{(x-1)^2 \ln x}{(x-1)^2+y^2}\right| \le \frac{(x-1)^2}{(x-1)^2+y^2}|\ln x| \le |\ln x|$

as $\lim\limits_{(x,y)\to(1,0)} |\ln x| = 0$, we can know $\lim\limits_{(x,y)\to(1,0)} \left|\frac{(x-1)^2 \ln x}{(x-1)^2+y^2}\right| = 0$.

Another way to find the limit is the polar coordinate

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

eg: Find the limit $\lim\limits_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2}$ by using polar coordinates

$$\begin{cases} \lim\limits_{(x,y)\to 0} \frac{x^3+y^3}{x^2+y^2} = \lim\limits_{r\to 0} r(\cos^3\theta+\sin^3\theta) \le \lim\limits_{r\to 0} r = 0 \\ \text{or we can compute directly} \\ \lim\limits_{(x,y)\to 0} \left|\frac{x^3+y^3}{x^2+y^2}\right| \le \lim\limits_{(x,y)\to 0} \frac{(x^2+y^2)(|x|+|y|)}{x^2+y^2} \le \lim\limits_{(x,y)\to 0} (|x|+|y|) = 0. \end{cases}$$

eg: show that $\lim\limits_{(x,y)\to(0,0)} \frac{x^2+xy}{2(x^2+y^2)}$

$\lim\limits_{(x,y)\to(0,0)} \frac{x^2+xy}{2(x^2+y^2)} \le \lim\limits_{(x,y)\to(0,0)} \frac{\cos^2\theta+\cos\theta\sin\theta}{2}$, so the limit do not exist!

Why? $\lim\limits_{\substack{(x,y)\to(0,0) \\ x=y}} \frac{2x^2}{4x^2} = \frac{1}{2}$ $\quad \lim\limits_{\substack{(x,y)\to 0 \\ \text{through} \\ (0,y)\to(0,0)}} \frac{0}{2y^2} = 0$, so the limit does not exist!

Consequences:

eg: find $\lim\limits_{(x,y)\to(0,0)} xy\ln(x^2+y^2)$

Solution. $\lim\limits_{(x,y)\to(0,0)} xy\ln(x^2+y^2) = \lim\limits_{r\to 0} r^2 \ln r^2 \cos\theta\sin\theta \le \lim\limits_{r\to 0} r^2 \ln r^2 = 0$

After computating, what is the advantage of the polar coordinate. with the help of the polar coordinate. Some times we only need to compute one-dimenzon limit.

The limit $(x,y)\to(a,b)$ is very very complex, because we have infite ways as $(x,y)\to(a,b)$.

(1) $\lim\limits_{x\to a}\lim\limits_{y\to b} f(x,y) \overset{\text{def}}{=\!=} \lim\limits_{x\to a}\Big(\underbrace{\lim\limits_{y\to b} f(x,y)}\Big)$

we take this limit first, we call this iterated limit.

Consider $f(x,y) = \frac{x+y}{x-y}$, $\lim\limits_{(x,y)\to(0,0)} \frac{x+y}{x-y}$ does not exist, it very clear. $(x,y)\to(0,0)$ as $(x,0)\to(0,0)$ and $(x,y)\to(0,0)$ as $(0,y)\to(0,0)$

But the iterated limit $\lim\limits_{x\to 0}\lim\limits_{y\to 0} \frac{x+y}{x-y} = 1$ and $\lim\limits_{y\to 0}\lim\limits_{x\to 0} \frac{x+y}{x+y} = -1$.

(2) The iterated limit is much weaker than the double limit. even we can know the iterated limit exists and equal, we can not say the double limit exists

$\lim\limits_{x\to 0}\lim\limits_{y\to 0} \frac{xy}{x^2+y^2} = 0$ and $\lim\limits_{y\to 0}\lim\limits_{x\to 0} \frac{xy}{x^2+y^2} = 0$ but $\lim\limits_{\substack{(x,y)\to(0,0)\\ \text{as } x=y}} \frac{xy}{x^2+y^2} = \lim\limits_{\substack{(x,y)\to(0,0)\\ \text{as } x=y}} \frac{x^2}{2x^2} = \lim\limits_{\substack{(x,y)\to(0,0)\\ \text{as } x=y}} \frac{1}{2}$

When you learn the single-variable calculus, you know the limit concept is prepared for the continuity and differentiable, etc, so we introduce

Def: Let $f: A \subseteq \mathbb{R}^n \to \mathbb{R}$ & $\vec{a} \in A$, then f is said to be continuous at $\vec{a}$ if $\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = f(\vec{a})$.

Equivalently, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that if $\vec{x} \in A$ & $\|\vec{x} - \vec{a}\| < \delta$, then $|f(\vec{x}) - f(\vec{a})| < \varepsilon$.

If $f: A \to \mathbb{R}$ is said to be continuous (on A) if f is continuous at every point in A.

Theorem: If $f, g: A \subset \mathbb{R}^n \to \mathbb{R}$ are continuous at $\vec{a} \in A$, then we have

(1) $f(\vec{x}) \pm g(\vec{x})$, $kf(\vec{x})$, $f(\vec{x})g(\vec{x})$ are continuous at $\vec{a}$, where k is a constant

(2) $\dfrac{f(\vec{x})}{g(\vec{x})}$ is continuous provided $g(\vec{a}) \neq 0$

Proof: $$\frac{f(\vec{a}+\vec{h})}{g(\vec{a}+\vec{h})} - \frac{f(\vec{a})}{g(\vec{a})} = \frac{f(\vec{a}+\vec{h})g(\vec{a}) - f(\vec{a})g(\vec{a}+\vec{h})}{g(\vec{a}+\vec{h})g(\vec{a})}.$$

We can ensure $g(\vec{h}+\vec{a}) \neq 0$ in a suitable neighbourhood.

and $$\lim_{\vec{h} \to 0}\left[f(\vec{a}+\vec{h})g(\vec{a}) - f(\vec{a})g(\vec{a}+\vec{h})\right] = 0.$$

Consequences:

(i) All polynomials of multi-variables are continuous (on R^n).

(ii) All rational functions of multi-variables are continuous on their domain of definition

(rational function $\overset{def}{=\!=} \frac{P(\vec{x})}{Q(\vec{x})}$ for some polynomials $P(\vec{x})$ & $Q(\vec{x})$

"domain of definition" of $\frac{P(\vec{x})}{Q(\vec{x})} = R^n \setminus \{\vec{x} : Q(\vec{x}) = 0\}$.)

egs: $\frac{x^3+y^2+yz}{x^2+y^2}$

How to prove this polynomial is continuous? We assume $x^2+y^2 \neq 0$.

Here we prove x^3+y^2+yz is continuous. For a give points (x_0, y_0, z_0), we want to show we have a δ_ε for any $\varepsilon > 0$. we have the $\underbrace{\sqrt{|x-x_0|^2+|y-y_0|^2+|z-z_0|^2} < \delta}_{\text{condition}}$, under the condition

we want to show $|x^3+y^2+yz - (x_0^3+y_0^2+y_0z_0)| < \varepsilon$

$$\Rightarrow |x^3-x_0^3+y^2-y_0^2+yz-y_0z_0| \le |(x-x_0)(x^2+xx_0+x_0^2)| + |(y-y_0)(y+y_0)| + |y||z-z_0| + |y-y_0||z_0|$$

$$\le |x-x_0|\left[(x-x_0+x_0)^2 + (x-x_0+x_0)x_0 + x_0^2\right] + |y-y_0||y+y_0| + |y-y_0+y_0||z-z_0| + |y-y_0||z_0|$$

$$\le \delta\left[(|\delta|+|x_0|)^2 + (\delta+|x_0|)|x_0| + |x_0|^2\right] + |\delta(2|y_0|+\delta)| + (\delta+|y_0|)\delta$$

$$\le C_{x_0,y_0,z_0}\delta$$

By selecting δ suitably small, we have ε small enough.

Fact: Let $\vec{a}$ be a zero of polynomial $Q(\vec{x})$ (i.e. $Q(\vec{a})=0$)

Then the rational function $r(\vec{x}) = \frac{P(\vec{x})}{Q(\vec{x})}$ can be "extended to a function continuous at $\vec{a}$ $\Leftrightarrow \lim_{\vec{x}\to\vec{a}} r(\vec{x})$ exists".

(1) $f(x,y) = \frac{x^4-y^4}{x^4+y^4}$.

$$\lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2} = \lim_{r\to 0} \frac{r^4(\cos^4\theta-\sin^4\theta)}{r^2} = \lim_{r\to 0} r^2\left(\cos^4\theta-\sin^4\theta\right)=0.$$

In fact $g(x,y) = \begin{cases} \frac{x^4-y^4}{x^2+y^2} & \text{if } (x,y)\neq(0,0), \\ 0 & \text{if } (x,y)=(0,0). \end{cases}$

Theorem. If • $f: A\subset\mathbb{R}^n\to\mathbb{R}$ is continuous at $\vec{a}\in A$

• g is a 1-variable function continuous at $f(\vec{a})$.

Then $g\circ f(\vec{x}) = g(f(\vec{x}))$ is continuous at $\vec{a}$, and

$$\lim_{\vec{x}\to\vec{a}} g(f(\vec{x})) = g(\lim_{\vec{x}\to\vec{a}} f(\vec{x})) = g(f(\vec{a})).$$

Interchange of the limit.

If: 1. $\lim_{x\to 0} g(x)=L$ exists, and $\Rightarrow$ $|x-0|<\delta_1 \Rightarrow \|g(x)-L\|<\varepsilon_1$

2. $\|\vec{x}-\vec{a}\|<\delta_2 \Rightarrow \|g(\vec{x})-g(\vec{a})\|<\varepsilon_2$.

(1) $\lim_{\vec{x}\to\vec{a}} g(f(\vec{x}))-g(f(\vec{a}))=0 \Rightarrow$ we have $\|\vec{x}-\vec{a}\|<\delta_1 \Rightarrow \|f(\vec{x})-f(\vec{a})\|<\delta_2 \Rightarrow \|g(f(\vec{x}))-g(f(\vec{a}))\|<\varepsilon$!.

Partial Derivatives

Def: Let $\Omega \subseteq \mathbb{R}^n$ be open

$\cdot f: \Omega \to \mathbb{R}^n$ be a function

Then the i-th partial derivative of f at $\vec{x}=(x_1,\cdots,x_n)\in\Omega$ is defined by

$$\frac{\partial f}{\partial x_i}(\vec{x})=\frac{\partial f}{\partial x_i}(x_1,\cdots,x_n)=\lim_{h\to 0}\frac{f(x_1,\cdots,x_i+h,x_n)-f(x_1,\cdots,x_n)}{h}.$$

eg. $f(x,y)=x^2+y^2$. $\frac{\partial f(x,y)}{\partial x}=2x$. $\frac{\partial f(x,y)}{\partial y}=2y$.

eg: $f(x,y,z)=xy^2-\cos(xz)$. $f_x=\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}(xy^2-\cos(xz))=y^2+z\sin(xz)$,

$$f_y=\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}(xy^2-\cos(xz))=2xy,$$

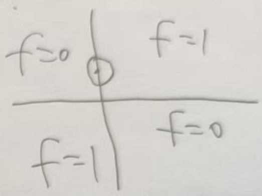
$$f_z=\frac{\partial f}{\partial z}=\frac{\partial}{\partial z}(xy^2-\cos(xz))=x\sin(xz).$$

eg: $f(x,y)=\begin{cases}1, & \text{if } xy\geq 0,\\ 0, & \text{if } xy<0.\end{cases}$

For $(1,1)$, $\lim_{h\to 0}\frac{f(1+h,1)-f(1,1)}{h}=\lim_{h\to 0}\frac{H}{h}=0.$

For (0,0),

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{f(0+h,1)-}{h} - \lim_{h\to 0}\frac{1-1}{h} = 0.$$



For (0,1),

$$\lim_{\substack{h\to 0\\ h>0}}\frac{f(0+h,1)}{h} = \lim_{h\to 0}\frac{1-1}{h} = 0,$$

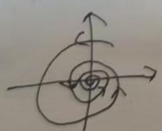
$$\lim_{\substack{h\to 0\\ h<0}}\frac{f(0+h,1)}{h} = \lim_{h\to 0}\frac{0-1}{h} = -\infty.$$

So $\frac{\partial f}{\partial x}(0,1)$ do not exist.

We also check $\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0}\frac{1-1}{h} = 0.$

$\lim_{(x,y)\to(0,0)} f(x)$ Do not existed

→ An oscillation sequence



Higher Order Partial Derivatives

$n=2$, $f(x,y) \Rightarrow \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}(x,y)\right), \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}f(x,y)\right), \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}f(x,y)\right), \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}f(x,y)\right).$

Notations: $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx}$, $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = (f_y)_x = f_{yx}$.

eg: $f(x,y) = x\sin y + y^2 e^{2x}$,

Then $\begin{cases} f_x = \sin y + 2y^2 e^{2x}, \\ f_y = x\cos y + 2y e^{2x}. \end{cases}$

$$\begin{cases} f_{xx} = (f_x)_x = 4y^2 e^{2x}, \\ f_{xy} = \cos y + 4y e^{2x}, \\ f_{yx} = \cos y + 4y e^{2x}, \\ f_{yy} = -x\sin y + 2e^{2x}. \end{cases}$$

eg: $$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

$\Rightarrow f_x = \frac{(x^2+y^2)(3x^2y - y^3) - xy(x^2-y^2)(2x)}{(x^2+y^2)^2}$ $(x,y) \neq (0,0)$

$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{f(h,0)}{h} = 0$

By definition,

$f_{xy}(0,0) = (f_x)_y(0,0) = \lim_{h\to 0} \frac{f_x(0,h) - f_x(0,0)}{h} = \lim_{h\to 0} \frac{-h^5}{h \cdot h^4} = -1.$

$$f_y(x,y) = \frac{(x^2+y^2)\,x(x^2-y^2) + (x^2+y^2)\,x(-2y) - xy(x^2-y^2)(2y)}{(x^2+y^2)^2}$$

$$= \frac{(x^2+y^2)\,x\,(x^2-y^2-2y) - xy(x^2-y^2)(2y)}{(x^2+y^2)^2}, \quad (x,y)\neq 0$$

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = 0.$$

$$f_y(h,0) = h$$

而 $f_{yx}(0,0) = (f_y)_x(0,0) = \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h} = 1$

So we have $f_{xy} \neq f_{yx}$.

When we can have $f_{xy} \neq f_{yx}$.

Theorem: Let $f: \Omega \to \mathbb{R}$ ($\Omega \subset \mathbb{R}^n$, open)

If f_{xy} & f_{yx} exist and are continuous on Ω, then

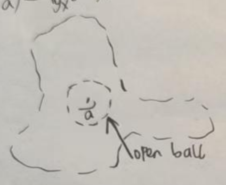
$$f_{xy} = f_{yx} \text{ on } \Omega.$$

Actually, we can a stronger version:

Theorem Let $\begin{cases} \cdot\ f: \Omega \to \mathbb{R} \ (\Omega \subset \mathbb{R}^n, \text{open}) \\ \cdot\ \vec{a} \in \Omega \end{cases}$

If $\begin{cases} \cdot\ f_{xy}\ \&\ f_{yx} \text{ exist in an open ball containing } \vec{a} \text{ and} \\ \cdot\ f_{xy}\ \&\ f_{yx} \text{ are continuous at } \vec{a} \end{cases}$

then $f_{xy}(\vec{a}) = f_{yx}(\vec{a})$.



Recall <u>mean value Theorem</u> for 1-variable function

let $f:[a,b]\to\mathbb{R}$, $\begin{cases} \cdot \text{ continuous on } [a,b]\ \& \\ \cdot \text{ differentiable on } (a,b) \end{cases}$

Then $\exists\ c\in(a,b)$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(c).$$