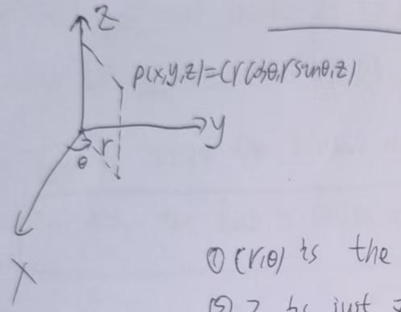
Other special coordinate, cylindrical coordinate

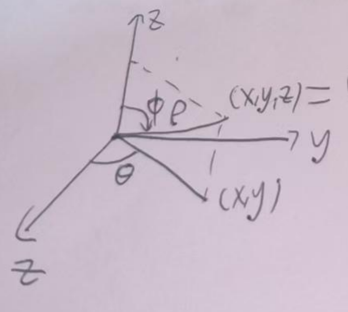


$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

$\begin{cases} x^2+y^2 \leq r^2 \\ 0 \leq z \leq y \end{cases}$

① (r,θ) is the polar coordinate for xy-plane
② z is just z-axis

special coordinates $\begin{cases} x = \rho\sin\phi\cos\theta, \\ y = \rho\sin\phi\sin\theta, \\ z = \rho\cos\phi. \end{cases}$ $\Rightarrow x^2+y^2+z^2 = r^2$



$(x,y,z) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$.

Some Topological concepts in R^n:

open ball: $B_R(x_0)=\{x\in R^n, |x-x_0|<R\}$ is the open ball of the radius R and centered at x_0.

Closed ball: $\overline{B_R(x_0)}=\{x\in R^n, |x-x_0|\le R\}$ is the closed ball of the radius R and centered at x_0.

① $\overline{B_R(x_0)}$ is called the closure of $B_R(x_0)$.

$\boxed{\text{If } n=2\text{, the ball is called the disk!}}$

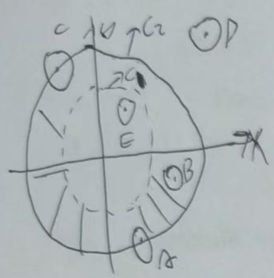
open ball　　　　Closed ball

The dotted line means not included

② For a closed ball $\overline{B_R(x_0)}$, $B_R(x_0)$ is called the interior of the $\overline{B_R(x_0)}$.

③

Example:



dot line means not included.

① C_1 and C_2 are boundaries

② A and C are boundary points, because some point are in this set but some others not.

③ B - Interior Points

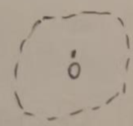
D - exterior points

E - exterior points

Now we can define the open set and closed set (Because we only have open balls and closed balls before).

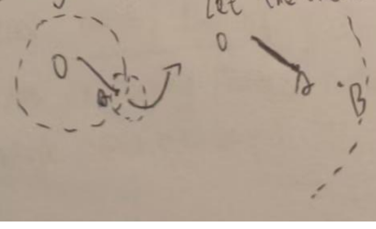
Def: A set $S \subset \mathbb{R}^n$ is called

(1) open if $\forall x \in S$, $\exists \delta > 0$ such that $B_\delta(\vec{x}) \subset S$



From this defination, we can find is we select one point, we can always find another point which is more far away from the original point.

Let the area be bigger



B is far away from the original point

Def: Let S be a set in $\mathbb{R}^n$, we define

(1) The interior of S is the set

$$Int(S) = \{\vec{x} \in \mathbb{R}^n : \exists \varepsilon > 0, s.t. B_\varepsilon(\vec{x}) \subset S\}$$

Points in $Int(S)$ are called interior points of

Q1: Is it possible that a set has no interior.

a single point $\{x\}$

(2) The ~~exterior~~ extension of S is the set

$$Ext(S) = \{\vec{x} \in \mathbb{R}^n : \exists \varepsilon > 0 \ s.t. B_\varepsilon(\vec{x}) \subset \mathbb{R}^n \setminus S\}$$

points in $Ext(S)$ are called extension points of S

Q2: Is it possible that a set has no exterior

(1) The whole space

(2) The rationals $\mathbb{Q}$ in $\mathbb{R}$

(3) The boundary of S is the set

$$\partial S = \{\vec{x} \in \mathbb{R}^n : \forall \varepsilon > 0 \ s.t. \ \begin{array}{l} B_\varepsilon(\vec{x}) \cap S \neq \phi, \& \\ B_\varepsilon(\vec{x}) \cap (\mathbb{R}^n \setminus S) \neq \phi. \end{array}$$

Points on ∂S are called Boundary Points of S.

① For set has no boundary points in Euclidean space, is it possible except whole space and empty set.
No.

② a point on the boundary of S belongs to S? No, open ball

(2) Closed if $R^n \setminus S$ is open

Equivalent definition

(1) S open $\Leftrightarrow S = \text{Int}(S)$

(2) S closed $\Leftrightarrow S = \text{Int}(S) \cup \partial S =$ Closure of S

ok we know some sets are open, some sets are closed, so every set is either closed or open?

The answer is No! There are some sets neither closed nor open, and some sets are closed and open.

$[\quad)$: neither closed nor open

R^N : Both closed and open

$$\begin{cases} R^N = \text{Int}(R^N) \\ R^N = \text{closure of } R^N \end{cases}$$

Now we should mention several sets.

Subset $S \subset R^2$	$B_1(0,0)$	$\overline{B_1(0,0)}$	$\partial B_1(0,0)$	R^2	ϕ (empty set)	
open or closed	open	closed	closed	open and closed	open and close	No open and No closed
					空的	

Def: Bound set, $S \subseteq R^n$ is called bounded if

$\exists\, m>0$ such that

$$S \subseteq B_m(\vec{0}) = \{\vec{x} \in R^n, \|\vec{x}\| < m\}$$

S is called <u>unbounded</u> if it is not bounded.

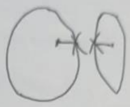
Question: Does a unbounded set have a boundary?

Yes

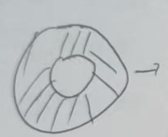
Def: $S \subset R^n$ is called Path-connected if any two pints in S can be connected by a curve in S;

Path - Connected

non-path-connected



anoth definition: Simple - Connected : any closed curve can shrink to one point.



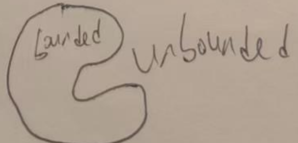
→ path- Connected but not simply Connected

After introducing the concept of the path-connected curve,

Thm (Jordan Curve Theorem)

A simple closed curve in R^2 divides R^2 into 2 path - connected component, with one bounded and one unbounded

no intersection except end point



Example: $h(x,y) = \cos(2\pi(x^2+y^2))$, $U = R^2$

$$L_c = \{(x,y) \in R^2;\ h(x,y) = c\}$$

$$= \{(x,y) \in R^2 : \cos(2\pi(x^2+y^2)) = c\}$$

Case 1: If $|c| > 1$, then $L_c = \phi$

Case 2: If $|c| \leq 1$, then $L_c = \{(x,y) \in R^2 : x^2+y^2 = \frac{1}{4\pi}\cos^{-1}(c)\}$

Case 2-1: If $\cos^{-1}(c) < 0$, $L_c = \phi$

Case 2-2: If $\cos^{-1}(c) = 0$, $L_c = \{(0,0)\}$

Case 2-3: If $\cos^{-1}(c) > 0$, $L_c = \{x^2+y^2 = \frac{1}{2\pi}\cos^{-1}(c)\}$

Limit of multi-variable functions

We define f as:

$$f : A \to R^m, \quad A \subseteq R^n$$

Def (ε-δ): Let $\vec{f} : A \subset R^n \to R^m$, $\vec{a} \in \bar{A} = A \cup \partial A$

we say that $\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = \vec{L}$. if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$\vec{x} \in A$ and $0 < ||\vec{x} - \vec{a}|| < \delta \Rightarrow ||\vec{f}(\vec{x}) - \vec{L}|| < \varepsilon$.

Remarks (i) $||\vec{x} - \vec{a}||$ = distance between $\vec{x}$ and $\vec{a}$ in R^n

(ii) $||\vec{f}(\vec{x}) - \vec{L}||$ means the distance between $\vec{f}(\vec{x})$ and $\vec{L}$ in R^m.

Proposition, limit of the componet

Prop: Let $\cdot\ A \subseteq R^n$

$\cdot\ \vec{a} \in \bar{A} = A \cup \partial A$

$\cdot\ \vec{f} = A \to \mathbb{R}^m$ with

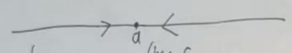
$$\vec{f}(\vec{x}) = \begin{bmatrix} f_1(\vec{x}) \\ \vdots \\ f_m(\vec{x}) \end{bmatrix} = \begin{bmatrix} f_1(x_1, \cdots, x_n) \\ \vdots \\ f_m(x_1, \cdots, x_n) \end{bmatrix}$$

Then $\lim\limits_{\vec{x}\to\vec{a}} \vec{f}(\vec{x}) = \vec{L} = \begin{bmatrix} \ell_1 \\ \vdots \\ \ell_m \end{bmatrix} \Leftrightarrow \lim\limits_{\vec{x}\to\vec{a}} f_i(\vec{x}) = \ell_i, \ \forall i = 1, 2, \cdots, m.$

From this theorem, we only need to focus on each componet of the vector valued function.

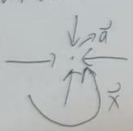
$$\vec{f}(x,y) = \begin{bmatrix} x+y \\ x^2+y^2+1 \end{bmatrix}, \quad \lim_{(x,y)\to(1,2)} \vec{f}(x,y) = \begin{bmatrix} \lim\limits_{(x,y)\to(1,2)} (x+y) \\ \lim\limits_{(x,y)\to(1,2)} (x^2+y^2+1) \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}.$$

Limit along a path

Recall : In one Variable, 

$$\lim_{x\to a} f(x) \Leftrightarrow \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$$

But for n-variables, $n \geq 2$, then situation is totally different.



There are many way for the process $\vec{x} \to \vec{a}$.

The $\lim\limits_{\vec{x}\to\vec{a}} f(\vec{x}) = L$ exists if and only if for any paths $\vec{x}\to\vec{a}$, the limit exists and equal. One may meet a situation that the limit exists in two different situation and they are not equal.

eg. $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$

Along the path $\lim_{(x,0)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}=1$

Along the path $\lim_{(0,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \frac{-y^2}{y^2}=-1$.

Along the path $\lim_{(x,x)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}=0$.

This means we have different values through different paths.

Example: (1) Consider $\lim_{(x,y)\to(1,2)} \frac{xy-2x-y+2}{(x-1)^2+(y-2)^2}$

$$\lim_{(x,y)\to(1,2)} \frac{xy-2x-y+2}{(x-1)^2+(y-2)^2} \Rightarrow \lim_{(1,y)\to(1,2)} 0=0$$

(2) along the line with slope $=m$ & passing through $(1,2)$

$$\lim_{\substack{(x,y)\to(1,2)\\ y-2=m(x-1)}} \frac{xy-2x-y+2}{(x-1)^2+(y-2)^2} = \lim_{\substack{(x,y)\to(1,2)\\ y-2=m(x-1)}} \frac{(x-1)(y-2)}{(x-1)^2+(y-2)^2} = \lim_{\substack{(x,y)\to(1,2)\\ y-2=m(x-1)}} \frac{m(x-1)^2}{(x-1)^2+m^2(x-1)^2} = \frac{m}{1+m^2}.$$