MATH 2010 F ADVANCED CALCULUS I

① We assume a function $f(x,y,z\cdots)$ and $f(x_1,x_2,\cdots x_n)$ as a function from $R^N \to R$.

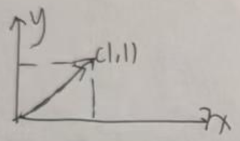
$$f(x,y)=x^2+y^2, \quad f(x,y,z)=3x^2+4y^2+5z^2, \quad f(x_1,x_2,\cdots,x_n)=x_1^2+\cdots+x_n^2$$

② We consider function from $R^N \to R$, so we must introduce a concept, vector.
a vector in R^N is like:

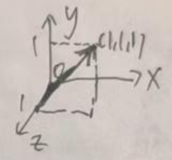
$$\vec{x}=(x_1,\cdots\cdots,x_n)\in R^n$$

we consider cartesian coordinate in the most cases:

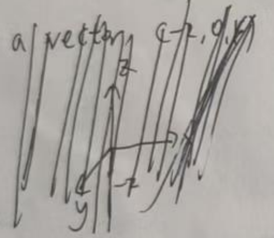
$n=2$



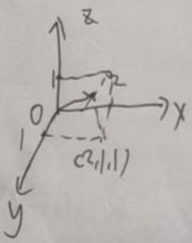
$n=3$



a ~~vector~~



a vector (2,1,1)



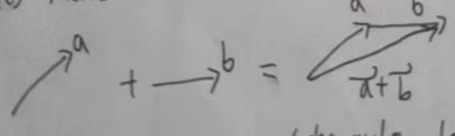
③ For vectors, if we have two vectors, how shall we consider interactions between them? We introduce

not a serious concept, for two vectors $\vec{a}=(a_1,\cdots,a_n)$ and $\vec{b}=(b_1,\cdots,b_n)$

(1) Equality $\vec{a}=\vec{b}$; $\vec{a_1}=\vec{b_1}$, $\vec{a_2}=\vec{b_2}$, $\vec{a_3}=\vec{b_3},\cdots\cdots$

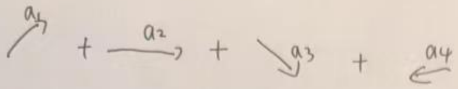
(2) Addition $\vec{a}+\vec{b}=(a_1+b_1, a_2+b_2,\cdots\cdots)$

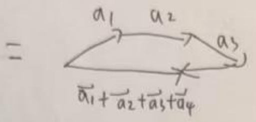
This means we add each component of them



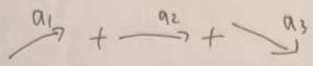
This is the triangular law.

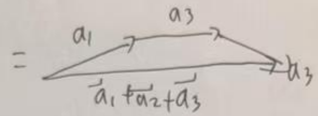
For many vectors,





= From head to the tail

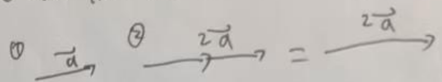




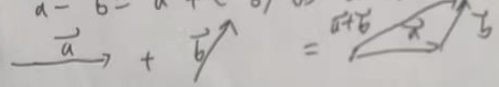
(3) Scalar multiplication

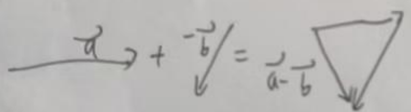
$$r\vec{a} = (ra_1, ra_2, ra_3, \cdots, ra_n)$$

we just put the factor "r" before each component of the vector.



(4) Subtraction: we just consider
$\vec{a}-\vec{b} = \vec{a}+(-\vec{b})$ as a special case of the addition.





(5) dot product:

Basic definition: $\vec{a}\cdot\vec{b} = (a_1,\cdots,a_n)\cdot(b_1,\cdots,b_n)$
$$= \sum_{i=1}^{n} a_i b_i.$$

The sum of the production of the each component.

For a special case:

$$\vec{a}\cdot\vec{a} = a_1^2+\cdots+a_n^2 = \|\vec{a}\|^2$$

We call $\|\vec{a}\|$ the length/magnitude of the vector.

Dot Product is a very important property

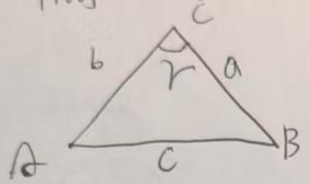
Distribution Law (1) $(\vec{a}+\vec{b})\cdot\vec{c} = \vec{a}\cdot\vec{c}+\vec{b}\cdot\vec{c}$

(2) $(r\vec{a})\cdot\vec{b} = \vec{a}\cdot(r\vec{b}) = r\vec{a}\cdot\vec{b}$

Commutative Property (3) $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$

Geometric Property (4) $\vec{a}\cdot\vec{b} = \|\vec{a}\|\,\|\vec{b}\|\cos\theta(\vec{a},\vec{b})$

Proof: the law of Cosines



$b = \|\overrightarrow{AC}\|,$
$a = \|\overrightarrow{BC}\|,$
$c = \|\overrightarrow{AB}\|.$

$$c^2 = a^2+b^2-2ab\cos\gamma$$
$$\Rightarrow (\overrightarrow{AB})^2 = (\overrightarrow{BC})^2+(\overrightarrow{CA})^2-2ab\cos\gamma$$
$$\Rightarrow (\overrightarrow{AC}+\overrightarrow{CB})^2 = (\overrightarrow{BC})^2+(\overrightarrow{CA})^2-2ab\cos\gamma$$
$$\Rightarrow \overrightarrow{AC}^2+\overrightarrow{BC}^2+2\overrightarrow{AC}\cdot\overrightarrow{CB} = (\overrightarrow{BC})^2+(\overrightarrow{AC})^2-2ab\cos\gamma$$
$$\Rightarrow \quad -2\overrightarrow{AC}\cdot\overrightarrow{CB} = 2ab\cos\gamma$$
$$\Rightarrow \quad \overrightarrow{CA}\cdot\overrightarrow{CB} = ab\cos\gamma$$

This gives the geometric property of the inner product.

Application of properties of the vector:

eg: suppose $\vec{v}$ & $\vec{w}$ are vectors of the same length. Show that $(\vec{v}+\vec{w})\cdot(\vec{v}-\vec{w})=0$.

Proof: $(\vec{v}+\vec{w})\cdot(\vec{v}-\vec{w}) = \vec{v}\cdot\vec{v} - \vec{v}\cdot\vec{w} + \vec{w}\cdot\vec{v} - \vec{w}\cdot\vec{w}$

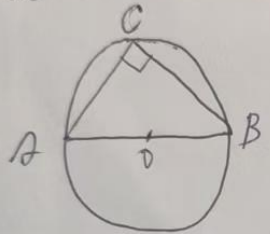
$= \|\vec{v}\|^2 - \|\vec{w}\|^2 = 0$

because $\vec{v}$ & $\vec{w}$ are the same length.

Geometric meaning:

Diagonal of the rhombus are perpendicular.

eg: AB is the diameter, C on the circle, show that $\angle ACB = 90°$.



they are

Aim to show that $\overrightarrow{AC}\cdot\overrightarrow{BC}=0$, to get the perpendicular.

$$\begin{cases} \overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC} \\ \overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC} \end{cases}$$

$$\overrightarrow{AC}\cdot\overrightarrow{BC} = (\overrightarrow{AO}+\overrightarrow{OC})\cdot(\overrightarrow{BO}+\overrightarrow{OC})$$

$$= \overrightarrow{AO}\cdot\overrightarrow{BO} + \overrightarrow{AO}\cdot\overrightarrow{OC} + \overrightarrow{OC}\cdot\overrightarrow{BO} + (\overrightarrow{OC})^2$$

$$= -\overrightarrow{AO}^2 + \overrightarrow{AO}\cdot\overrightarrow{OC} + \overrightarrow{BO}\cdot\overrightarrow{OC} + \underbrace{(\overrightarrow{OC})^2}_{r^2}$$

$$= -\overrightarrow{BO}\cdot\overrightarrow{OC} + \overrightarrow{BO}\cdot\overrightarrow{OC}$$

$$= 0.$$

Special structure of R^3: Cross Product

Two inequalities:

① Cauchy – Schwartz Inequality:

Let $\vec{a}, \vec{b} \in \mathbb{R}^n$. Then $|\vec{a} \cdot \vec{b}| \leq \|\vec{a}\| \, \|\vec{b}\|$

i.e. $\left|\sum_{i=1}^{n} a_i b_i\right| \leq \sqrt{\sum_{i=1}^{n} a_i^2} \sqrt{\sum_{i=1}^{n} b_i^2}$

② Equality holds under conditions:

$$\vec{a} = r\vec{b} \text{ or } \vec{b} = r\vec{a}$$

for some $r \in R$.

Proof: ① $\vec{a}$ or $\vec{b}$ is zero, trivial.

② $|\vec{a} \cdot \vec{b}| = \|a\| \, \|b\| \, |\cos\theta\langle \vec{a}, \vec{b}\rangle|$

$\cos\theta\langle\vec{a}, \vec{b}\rangle = 1$ only occur under conditions $\vec{a} = r\vec{b}$.

Triangle Inequality

Let $\vec{a}, \vec{b} \in R^n$. Then

$$\|\vec{a} + \vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|$$

Equality holds $\Leftrightarrow \vec{a} = r\vec{b}$ or $\vec{b} = r\vec{a}$, for some $r \geq 0$.

Proof: $\|\vec{a}+\vec{b}\|^2 = (\vec{a}+\vec{b}) \cdot (\vec{a}+\vec{b}) = \|\vec{a}\|^2 + \|\vec{b}\|^2 + 2\vec{a} \cdot \vec{b}$.

and $(\|\vec{a}\| + \|\vec{b}\|)^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 + 2\|\vec{a}\| \, \|\vec{b}\|$

Equality holds $\Leftrightarrow \vec{a} \cdot \vec{b} = \|\vec{a}\| \, \|\vec{b}\|$

$\Leftrightarrow$ Equality holds for Cauchy-Sc

$\Leftrightarrow \vec{a} = r\vec{b}$ or $\vec{b} = r\vec{a}$.

Special structure of R^3: Cross Product

Let $\vec{a}=(a_1,a_2,a_3)$, $\vec{b}=(b_1,b_2,b_3)\in R^3$

Then we have the cross product

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$ $\to$ unit vector in R^3

$$=(-1)^{1+1}\hat{i}\begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix}+(-1)^{1+2}\hat{j}\begin{vmatrix}a_1 & a_3\\ b_1 & b_3\end{vmatrix}+(-1)^{1+3}\hat{k}\begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}$$

Here I just think you know

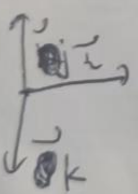
Example: let $\vec{a}=(2,3,5)$, $\vec{b}=(1,2,3)$, we have

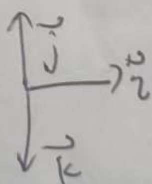
$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & 5\\ 1 & 2 & 3\end{vmatrix}=(-1)^{1+1}\hat{i}\begin{vmatrix}3 & 5\\ 2 & 3\end{vmatrix}+(-1)^{1+2}\hat{j}\begin{vmatrix}2 & 5\\ 1 & 3\end{vmatrix}+(-1)^{1+3}\begin{vmatrix}2 & 3\\ 1 & 2\end{vmatrix}\hat{k}$$

$$=-\hat{i}-\hat{j}+\hat{k}=(-1,-1,1).$$

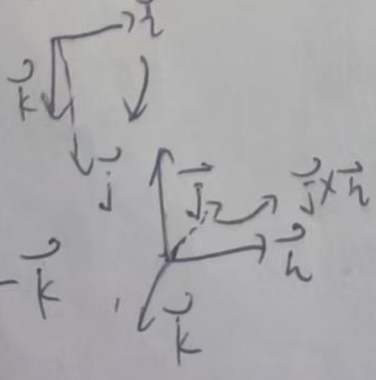
Some results of the dot Product

① $\hat{i}\times\hat{i}=0$, ② $\hat{i}\times\hat{j}=\hat{k}$, ③ $\hat{i}\times\hat{k}=-\hat{j}$





Use your hand (right)!



④ $\vec{j}\times\vec{i}=-\vec{k}$,

Properties of Cross product

(1) $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$ easily from the right-hand rule

distribution law (2) $(\alpha\vec{a}+\beta\vec{b})\times\vec{c} = \alpha\vec{a}\times\vec{c}+\beta\vec{b}\times\vec{c}$

$\vec{a}\times(\alpha\vec{b}+\beta\vec{c}) = \alpha\vec{a}\times\vec{b}+\beta\vec{a}\times\vec{c}$

(3) $(\vec{a}\times\vec{b})\perp\vec{a}$ and $(\vec{a}\times\vec{b})\perp\vec{b}$ easily from the definition

Geometric property (4) $\|\vec{a}\times\vec{b}\| = \|\vec{a}\|\,\|\vec{b}\|\sin\theta$

$$\vec{a}\times\vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (-1)^{1+1} i \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}$$

$$+(-1)^{1+2} j \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + (-1)^{1+3} k \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= (a_2b_3-b_2a_3)\,i - (a_1b_3-a_3b_1)\,j + (a_1b_2-a_2b_1)\,k$$

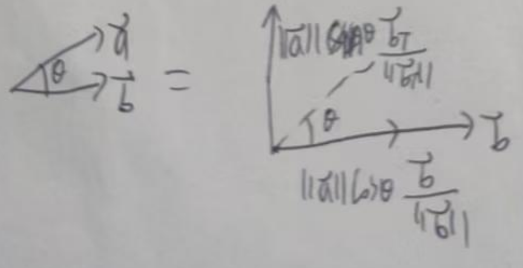
$$\|\vec{a}\times\vec{b}\|^2 = (a_2b_3-b_2a_3)^2+(a_1b_3-a_3b_1)^2+(a_1b_2-a_2b_1)^2$$

$$= (a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)-(a_1b_1+a_2b_2+a_3b_3)^2$$

$$= \|\vec{a}\|^2\|\vec{b}\|^2-(\vec{a}\cdot\vec{b})^2$$

$$= \|\vec{a}\|^2\|\vec{b}\|^2\sin^2\theta$$

Explanation



$$\Rightarrow \vec{a}\times\vec{b} = \left(\|\vec{a}\|\sin\theta\frac{\vec{b}_T}{\|\vec{b}_T\|}+\|\vec{a}\|\cos\theta\frac{\vec{b}}{\|\vec{b}\|}\right)\times\vec{b} = \|\vec{a}\|\sin\theta\frac{\vec{b}_T}{\|\vec{b}_T\|}\times\vec{b} \quad \checkmark$$

$$= \|\vec{a}\|\,\|\vec{b}\|\sin\theta\frac{\vec{b}_T}{\|\vec{b}_T\|}\times\frac{\vec{b}}{\|\vec{b}\|}$$

meaning of $\frac{1}{2}\vec{a}\times\vec{b}$, the area of the triangular