MATH2010

Tutorial 6

Exercises 13.3

Q69 Find the value of $\frac{\partial z}{\partial x}$ at the point $(1,1,1)$ if the equation $xy + z^3x - 2yz = 0$ defines z as a function of the two independent variables x and y and the partial derivative exists.

Differentiating Implicitly (with respect to x), ($z = z(x,y)$, regard y as constant)

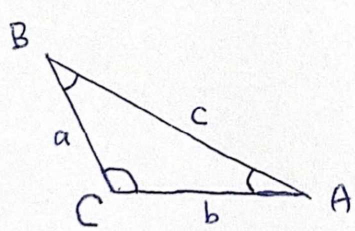
$$y + z^3 \cdot 1 + 3z^2 \frac{\partial z}{\partial x} \cdot x - 2y \frac{\partial z}{\partial x} = 0$$

$$(3xz^2 - 2y)\frac{\partial z}{\partial x} = -y - z^3$$

$$\frac{\partial z}{\partial x} = \frac{-y - z^3}{3xz^2 - 2y}$$

At $(1,1,1)$, $\frac{\partial z}{\partial x} = \frac{-1-1}{3-2} = -2$

Q71



Express A implicitly as a function of a, b, and c and calculate $\frac{\partial A}{\partial a}$ and $\frac{\partial A}{\partial b}$.

By the Cosine Law, $a^2 = b^2 + c^2 - 2bc\cos A$.

Differentiating Implicitly with respect to a,

$$2a = -2bc(-\sin A)\frac{\partial A}{\partial a} = 2bc\sin A\frac{\partial A}{\partial a}$$

$$\Rightarrow \frac{\partial A}{\partial a} = \frac{a}{bc\sin A}$$

Differentiating Implicitly with respect to b,

$$0 = 2b - \left(2c\cos A \cdot 1 + 2bc(-\sin A)\frac{\partial A}{\partial b}\right)$$

$$2c\cos A - 2b = 2bc\sin A\frac{\partial A}{\partial b}$$

$$\Rightarrow \frac{\partial A}{\partial b} = \frac{c\cos A - b}{bc\sin A}$$

Q73 Express v_x in terms of u and v if the equations $x = v\ln u$ and $y = u\ln v$ define u and v as functions of the independent variables x and y, and if v_x exists. (Hint: Differentiate both equations with respect to x and solve for v_x by eliminating u_x.)

$$x = v\ln u \Rightarrow 1 = v\frac{1}{u}u_x + v_x\ln u$$

$$y = u\ln v \Rightarrow 0 = u\frac{1}{v}v_x + u_x\ln v$$

$$\begin{cases} \left(\frac{v}{u}\right)u_x + (\ln u)v_x = 1 \\ (\ln v)u_x + \left(\frac{u}{v}\right)v_x = 0 \end{cases}$$

$$\begin{cases} \left(\frac{v}{u}\right)(\ln v)u_x + (\ln u)(\ln v)v_x = \ln v \\ \left(\frac{v}{u}\right)(\ln v)u_x + \left(\frac{v}{u}\right)\left(\frac{u}{v}\right)v_x = 0 \end{cases}$$

$$\ominus:\ ((\ln u)(\ln v) - 1)v_x = \ln v \Rightarrow v_x = \frac{\ln v}{(\ln u)(\ln v) - 1}$$

Q81 Let $f(x,y) = \begin{cases} y^3, & y \geq 0 \\ -y^2, & y < 0. \end{cases}$

Find f_x, f_y, f_{xy}, and f_{yx}, and state the domain for each partial derivative.

- $f_x(x,y) = 0$ for all points $(x,y) \in \mathbb{R}^2$ Domain $= \mathbb{R}^2$

- For f_y:

For $y > 0$, $f_y(x,y) = 3y^2$

For $y < 0$, $f_y(x,y) = -2y$

For $y = 0$, $f_y(x,0) = \lim_{h \to 0} \frac{f(x, 0+h) - f(x,0)}{h}$

$= \lim_{h \to 0} \frac{f(x,h) - 0}{h}$

$= \lim_{h \to 0} \frac{f(x,h)}{h}$

$\lim_{h \to 0^-} \frac{f(x,h)}{h} = \lim_{h \to 0^-} \frac{-h^2}{h} = \lim_{h \to 0^-} (-h) = 0,$

$\lim_{h \to 0^+} \frac{f(x,h)}{h} = \lim_{h \to 0^+} \frac{h^3}{h} = \lim_{h \to 0^+} h^2 = 0.$

$\Rightarrow f_y(x,0) = \lim_{h \to 0} \frac{f(x,h)}{h} = 0$

$\Rightarrow f_y(x,y) = \begin{cases} 3y^2, & y \geq 0 \\ -2y, & y < 0 \end{cases}$ for any $x \in \mathbb{R}$. Domain $= \mathbb{R}^2$

- $f_{xy}(x,y) = 0$ for all points $(x,y) \in \mathbb{R}^2$ Domain $= \mathbb{R}^2$

- $f_{yx}(x,y) = 0$ for all points $(x,y) \in \mathbb{R}^2$ Domain $= \mathbb{R}^2$

Q89 $f(x,y,z) = (x^2+y^2+z^2)^{-\frac{1}{2}}$

Show that the function satisfies a Laplace equation.

$$\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0 \right)$$

$$\frac{\partial f}{\partial x} = -\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}(2x) = -x(x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$\frac{\partial f}{\partial y} = -\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}(2y) = -y(x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$\frac{\partial f}{\partial z} = -\frac{1}{2}(x^2+y^2+z^2)^{-\frac{3}{2}}(2z) = -z(x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$\frac{\partial^2 f}{\partial x^2} = (-x)\left(-\frac{3}{2}(x^2+y^2+z^2)^{-\frac{5}{2}}(2x)\right) - 1\cdot(x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$= 3x^2(x^2+y^2+z^2)^{-\frac{5}{2}} - (x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$\frac{\partial^2 f}{\partial y^2} = 3y^2(x^2+y^2+z^2)^{-\frac{5}{2}} - (x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$\frac{\partial^2 f}{\partial z^2} = 3z^2(x^2+y^2+z^2)^{-\frac{5}{2}} - (x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$\Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 3(x^2+y^2+z^2)(x^2+y^2+z^2)^{-\frac{5}{2}} - 3(x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$= 3(x^2+y^2+z^2)^{-\frac{3}{2}} - 3(x^2+y^2+z^2)^{-\frac{3}{2}}$$

$$= 0$$