$f(2,\frac{1}{2},\frac{15}{8},0)=\frac{17}{64}$

$f(-2,-\frac{1}{2},\frac{-225}{136},\frac{15}{17})=\frac{17}{64}$

$\Rightarrow$ distance between $\mathcal{C}$ and $L=\frac{\sqrt{17}}{8}$

Implicit Function theorem

Recall: Implicit differentiation

eg. $x^2+y^2+z^2=2$ and if $z=z(x,y)$,

then $\begin{cases} \frac{\partial}{\partial x}(x^2+y^2+z^2)=0 \Rightarrow 2x+2z\frac{\partial z}{\partial x}=0, \\ \frac{\partial}{\partial y}(x^2+y^2+z^2)=0 \Rightarrow 2y+2z\frac{\partial z}{\partial y}=0. \end{cases}$

If the point (x,y,z) satisfies $z\neq 0$,

then we have $\frac{\partial z}{\partial x}=-\frac{x}{z}$ & $\frac{\partial z}{\partial y}=-\frac{y}{z}$.

Question: If a level set $g(x,y)=C$ is given, Can we "solve" the Constraint?

General Situation (in 3-variables)

If $z=z(x,y)$ (differentiable), the implicite differentiation

$\frac{\partial}{\partial x}$: $\frac{\partial F}{\partial x}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial x}=0$,

$\frac{\partial}{\partial y}$: $\frac{\partial F}{\partial y}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial y}=0$.

If $F(a)=c$ & $\frac{\partial F}{\partial z}(a) \neq 0$, then

$$\begin{bmatrix} \frac{\partial z}{\partial x} \\ \frac{\partial z}{\partial y} \end{bmatrix} = -\frac{1}{\frac{\partial F}{\partial z}(a)} \begin{bmatrix} \frac{\partial F}{\partial x}(a) \\ \frac{\partial F}{\partial y}(a) \end{bmatrix}.$$

Question: can we solve $\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} y(x) \\ z(x) \end{bmatrix}$?

observation: If we have $y=y(x)$ & $z=z(x)$

differentiation w.r.t. $x \Rightarrow \frac{d}{dx}\left(x^2+(y(x))^2+(z(x))^2\right)=0$

$$\Rightarrow 2x+2y\frac{dy}{dx}+2z\frac{dz}{dx}=0$$

$$\Rightarrow y\frac{dy}{dx}+z\frac{dz}{dx}=-x.$$

and we also have $\frac{d}{dx}(x+z(x))=0 \Rightarrow \frac{dz}{dx}=-1.$

$$\begin{bmatrix} y & z \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{dy}{dx} \\ \frac{dz}{dx} \end{bmatrix} = \begin{bmatrix} -x \\ 1 \end{bmatrix}.$$

If $\det \begin{bmatrix} y & z \\ 0 & 1 \end{bmatrix} \neq 0$, then one can solve

for $\begin{bmatrix} \frac{dy}{dx} \\ \frac{dz}{dx} \end{bmatrix} = \underbrace{\begin{bmatrix} y & z \\ 0 & 1 \end{bmatrix}^{-1}}_{\text{We need the determinant is non-zero}} \begin{bmatrix} -x \\ -1 \end{bmatrix}$

$(x,y,z)=(0,1,1)$

$$\begin{bmatrix} \frac{dy}{dx} \\ \frac{dz}{dx} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

<u>In general</u>, given $\mathcal{C}: \begin{cases} F_1(x,y,z)=C_1, \\ F_2(x,y,z)=C_2. \end{cases}$

Suppose $F_i(a,b,c)=C_i$, $i=1,2$

Assume $y=y(x)$, $z=z(x)$ near (a,b,c)

(Implicit differentiation) $\begin{cases} \frac{d}{dx} F_1(x,y,z)=0, \\ \frac{d}{dx} F_2(x,y,z)=0 \end{cases}$

$$\Rightarrow \begin{cases} \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y}\frac{dy}{dx} + \frac{\partial F_1}{\partial z}\frac{dz}{dx} = 0, \\ \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}\frac{dy}{dx} + \frac{\partial F_2}{\partial z}\frac{dz}{dx} = 0. \end{cases}$$

$$\begin{bmatrix} \frac{\partial F_1}{\partial x} \\ \frac{\partial F_2}{\partial x} \end{bmatrix} + \begin{bmatrix} \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \end{bmatrix} \begin{bmatrix} \frac{dy}{dx} \\ \frac{dz}{dx} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \end{bmatrix} \begin{bmatrix} \frac{dy}{dx} \\ \frac{dz}{dx} \end{bmatrix} = -\begin{bmatrix} \frac{\partial F_1}{\partial x} \\ \frac{\partial F_2}{\partial x} \end{bmatrix}$$

If $\det\begin{bmatrix} \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \end{bmatrix} \neq 0$ at (a,b,c), i.e. $\begin{bmatrix} \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \end{bmatrix}$ is invertible.

then $\begin{bmatrix} \frac{dy}{dx} \\ \frac{dz}{dx} \end{bmatrix} = -\begin{bmatrix} \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial F_1}{\partial x} \\ \frac{\partial F_2}{\partial x} \end{bmatrix}$ at (a,b,c).

Theorem (Implicit Function Theorem)

Let $U \subseteq \mathbb{R}^{n+k}$ be open, $\vec{F}: U \to \mathbb{R}^k$, $\vec{F} = \begin{bmatrix} F_1 \\ \vdots \\ F_k \end{bmatrix}$ be C^1. Denote $\vec{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$ & $\vec{y} = (y_1, \cdots, y_k) \in \mathbb{R}^k$

$$\vec{F}(\vec{x}, \vec{y}) = \begin{bmatrix} F_1(\vec{x}, \vec{y}) \\ \vdots \\ F_k(\vec{x}, \vec{y}) \end{bmatrix} = \begin{bmatrix} F_1(x_1, \cdots, x_n, y_1, \cdots, y_k) \\ \vdots \\ F_k(x_1, \cdots, x_n, y_1, \cdots, y_k) \end{bmatrix}$$

Suppose $(\vec{a}, \vec{b}) \in U$, where $\vec{a} \in \mathbb{R}^n$, $\vec{b} \in \mathbb{R}^k$ such that

$$\vec{F}(\vec{a}, \vec{b}) = \vec{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix} \in \mathbb{R}^k$$

and the $k \times k$ matrix

$$\left[\frac{\partial F^i}{\partial y_j}(\vec{a}, \vec{b}) \right] = \begin{bmatrix} \frac{\partial F_1}{\partial y_1}(\vec{a}, \vec{b}) & \cdots & \frac{\partial F_1}{\partial y_k}(\vec{a}, \vec{b}) \\ \vdots & & \vdots \\ \frac{\partial F_k}{\partial y_1}(\vec{a}, \vec{b}) & \cdots & \frac{\partial F_k}{\partial y_k}(\vec{a}, \vec{b}) \end{bmatrix}$$

is invertible

Then there are open set $U \subseteq \mathbb{R}^n$ containg $\vec{a}$, and $V \subseteq \mathbb{R}^k$ containing $\vec{b}$ such that there exists a unique function $\vec{\varphi} : V \to V$ with $\vec{\varphi}(\vec{a}) = \vec{b}$ and

$$\vec{F}(\vec{x}, \vec{\varphi}(\vec{x})) = \vec{c}, \quad \forall x \in V.$$

moreover, $\vec{\varphi}$ is C^1 and

$$\left[\frac{\partial \varphi_j}{\partial x_l}(\vec{x}) \right]_{k \times n} = - \left[\frac{\partial F^i}{\partial y_j}(\vec{x}, \vec{\varphi}(\vec{x})) \right]^{-1}_{k \times k} \left[\frac{\partial F_i}{\partial x_l}(\vec{x}, \vec{\varphi}(\vec{x})) \right]$$

$x^2+y^2+z^2=2$ Solve $z=z(x,y)$

$g(x,y,z)=x^2+y^2+z^2=2$ near $(0,1,1)$

$\frac{\partial g}{\partial z}(0,1,1)=2\neq 0$, By IFT,

$\exists\, z=z(x,y)$ "near" $(0,1)$

s.t. $\begin{cases} g(x,y,z(x,y))=2 \\ z(0,1)=1 \end{cases}$

$x^2+y^2+(z(x,y))^2=2$

$\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ can be computed by implicit differentiation.

Special (B) Case: $n=1, k=2$ (2 constraints)

$\vec{F}: \Omega\subset\mathbb{R}^{1+2}\to\mathbb{R}^2$

$$\vec{F}(x,y_1,y_2)=\begin{bmatrix} F_1(x,y_1,y_2) \\ F_2(x,y_1,y_2)\end{bmatrix}=\begin{bmatrix} c_1 \\ c_2\end{bmatrix}=\vec{c}.$$

suppose (a,b_1,b_2) satisfies the constraints $\vec{F}(a,b_1,b_2)=\vec{c}$,

then IFT means:

if $\begin{bmatrix} \frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} \\ \frac{\partial F_2}{\partial y_1} & \frac{\partial F_2}{\partial y_2}\end{bmatrix}(a,b_1,b_2)$ is invertible then $\exists y_1 = \varphi_1(x)$ & $y_2 = \varphi_2(x)$ "near" (a, b_1, b_2)

solving the constraints (locally)

$$\begin{cases} F_1(x, \varphi_1(x), \varphi_2(x)) = c_1, \\ F_2(x, \varphi_1(x), \varphi_2(x)) = c_2 \end{cases}$$

& $\begin{cases} \varphi_1(a) = b_1, \\ \varphi_2(a) = b_2. \end{cases}$

eg3 $\begin{cases} x^2+y^2+z^2=2 \\ x+z=1 \end{cases}$ solve for $y=y(x)$, $z=z(x)$?

near $(0,0,1)$.

$$\begin{cases} g(x,y,z) = x^2+y^2+z^2=2, \\ h(x,y,z) = x+z=1 \end{cases}$$

near $(0,0,1)$.

$$\begin{bmatrix} \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \\ \frac{\partial h}{\partial y} & \frac{\partial h}{\partial z} \end{bmatrix}_{(0,1,1)} = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$$

By IFT,

$\exists y = y(x)$, $z = z(x)$ "near" $(0,1,1)$ s.t.

$$\begin{cases} g(x, y(x), z(x)) = 2 \\ h(x, y(x), z(x)) = 1 \end{cases}$$

eg: Consider the constraints

$$\begin{cases} xz + \sin(yz - x^2) = 8, \\ x + 4y + 3z = 18 \end{cases}$$

$(2,1,4)$ is a solution.

Can we solve 2 of the variables as functions of the remaining variable?

Solution: $\vec{F}(x,y,z) = \begin{bmatrix} F_1(x,y,z) \\ F_2(x,y,z) \end{bmatrix} = \begin{bmatrix} xz + \sin(yz - x^2) \\ x + 4y + 3z \end{bmatrix}$

$$D\vec{F} = \begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \end{bmatrix} = \begin{bmatrix} z - 2x\cos(yz - x^2) & z\cos(yz - x^2) & x + y\cos(yz - x^2) \\ 1 & 4 & 3 \end{bmatrix}$$

$$D\vec{F}(2,1,4) = \begin{bmatrix} 0 & 4 & 3 \\ 1 & 4 & 3 \end{bmatrix}, \quad \det\begin{array}{c} \begin{array}{cc} y & z \\ \downarrow & \downarrow \end{array} \\ \begin{pmatrix} 4 & 3 \\ 4 & 3 \end{pmatrix} \end{array} = 0$$

<u>No conclusion</u> on whether y, z can be solved as functions of x near $(2,1,4)$.

Theorem (Inverse Function Theorem)

Let $\vec{f}: \Omega \to \mathbb{R}^n$ be C^1, ($\Omega \subset \mathbb{R}^n$, open)

Suppose $D\vec{f}(\vec{a})$ is invertible ($n\times n$ matrix)

Then $\exists$ open sets $U \subseteq \mathbb{R}^n$ containing $\vec{a}$,

$V \subseteq \mathbb{R}^n$ containing $\vec{b} = \vec{f}(\vec{a})$,

such that $\exists$ a unique function $\vec{g}: V \to U$ with $\vec{g}(\vec{b}) = \vec{a}$

satisfying
$$\begin{cases} \vec{g}(\vec{f}(\vec{x})) = \vec{x}, & \forall \vec{x} \in U \\ \vec{f}(\vec{g}(\vec{y})) = \vec{y}, & \forall \vec{y} \in V \end{cases}$$

moreover, $\vec{g}$ is C^1 and

$$D\vec{g}(\vec{y}) = [D\vec{f}(\vec{g}(\vec{y}))]^{-1}, \ \forall \vec{y} \in V.$$

Remark: $\vec{g} = (\vec{f}|_V)^{-1}$ is called a local inverse of $\vec{f}$ at $\vec{a}$.

eg: $\vec{f}$ is actually linear:
$$\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

clearly $\vec{f}$ is invertible $\Leftrightarrow$ $\begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & & \vdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix}$ invertible

$$\vec{g}(\vec{y}) = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & & \vdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix}^{-1} \left[\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} - \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \right] = f^{-1}$$

But
$$\begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & & \vdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix} = D\vec{f} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$
$\therefore$ the condition required by IFT is satisfied.

$$D\vec{g} = \begin{pmatrix} \frac{\partial g_1}{\partial y_1} & \cdots & \frac{\partial g_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_n}{\partial y_1} & \cdots & \frac{\partial g_n}{\partial y_n} \end{pmatrix} = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \cdots & \frac{\partial x_n}{\partial y_n} \end{pmatrix} = \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix}^{-1} = (D\vec{f})^{-1}.$$

eg: $\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$ $\vec{F}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x^2 - y^2 \\ 2xy \end{bmatrix}$

clearly $\vec{F}$ is not globally invertible: $\vec{F}\left(\begin{bmatrix} -x \\ -y \end{bmatrix}\right) = \vec{F}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$

To check this: $D\vec{F} = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}$.

$\det D\vec{F} = 4(x^2+y^2) \geq 0$ & "=0" only if $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

For $(x,y) \neq (0,0)$, IFT (Inverse Function Theorem)

$\Rightarrow \vec{F}$ has a local inverse at (x,y) $(\neq (0,0))$.

For instance, let $(x,y) = (1,-1)$ & $\vec{g}(u,v)$ be a local inverse of $f(x,y)$ "near" $(x,y) = (1,-1)$.

$\vec{F}(1,-1) = (0,-2) \Rightarrow \vec{g}(0,-2) = (1,-1)$

$D\vec{g}(0,-2) = (D\vec{F}(1,-1))^{-1} = \begin{pmatrix} 2 & 2 \\ -2 & 2 \end{pmatrix}^{-1} = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

Explicit Calculation of $\vec{g}(u,v)$:

$$\begin{cases} u = x^2 - y^2, \\ v = 2xy \end{cases}$$

near $(x,y)=(1,-1) \Rightarrow x \neq 0 \Rightarrow y = \frac{v}{2x}$

$$\Rightarrow u = x^2 - \left(\frac{v}{2x}\right)^2$$

$$\Rightarrow 4x^4 - 4ux^2 - v^2 = 0 \Rightarrow x^2 = \frac{4u \pm \sqrt{(-4u)^2 - 4\cdot 4(-v)^2}}{8} = \frac{u \pm \sqrt{u^2+v^2}}{2}$$

put $(x,y) = (1,-1) \Rightarrow (u,v) = (0,-2)$

$$1^2 = \frac{0 \pm \sqrt{0 \pm (-2)^2}}{2}$$

$$\Rightarrow x^2 = \frac{u + \sqrt{u^2+v^2}}{2} \Rightarrow x = \sqrt{\frac{u + \sqrt{u^2+v^2}}{2}}$$

& $y = \frac{v}{2x} = \frac{\sqrt{2}\,v}{2\sqrt{u + \sqrt{u^2+v^2}}}$

Remark: In Implicit Function Theorem & Inverse Function Theorem,

we need to check det. of Jacobian matrix (a submatrix) is nonzero.

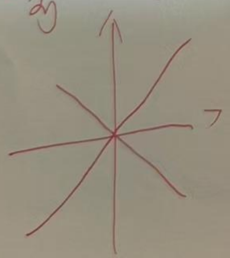
In case that the det = 0, we have

No conclusion:

eg: Implicit Function Theorem

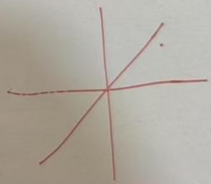
$F(x,y) = x^2 - y^2 = 0$

$\frac{\partial F}{\partial y} = -2y \big|_{(0,0)} = 0.$

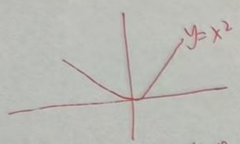


$F(x,y) = x^3 - y^3 = 0$

$\frac{\partial F}{\partial y} = -3y^2 \big|_{(0,0)} = 0$



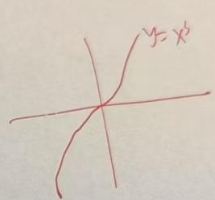
Inverse function Thm



No injective near $x=0$

$\Rightarrow$ No local inverse near $x=0$

$f(x)=x^3$



$g(y)=y^{\frac{1}{3}}$ is a local inverse near $x=$