(2) $\{y=0, 0\le x\le 1\}$: $f(x,0)=-4x^3$

$f(1,0)=-4\le f(x,0)\le 0=f(0,0)$

(3) $\{x=1, 0\le y\le 1\}$: $f(1,y)=6y-4-3y^2=-3(y-1)^2-1$

$f(1,0)=-4\le f(1,y)\le -1=f(1,1)$.

(4) $\{y=1, 0\le x\le 1\}$: $f(x,1)=6x-4x^3-3$

$0=\frac{d}{dx}f(x,1)=6-12x^2 \Rightarrow x=\frac{1}{\sqrt{2}}$

$f(\frac{1}{\sqrt{2}},1)=-(3-2\sqrt{2})<0$

$f(1,1)=-1$

$f(0,1)=-3,$

$f(0,1)=-3\le f(x,1)\le -(3-2\sqrt{2})=f(\frac{1}{\sqrt{2}},1)$

$\Rightarrow$

max: $f(0,0)=0$, min $f(1,0)=-4$.

Compare values

global max at $(\frac{1}{2},\frac{1}{2})$ with value $\frac{1}{4}$

global min at $(1,0)$ with value $-4$

第　　题（答题不得写在红线外）

Matrix form for $2^{nd}$ order Taylor polynomial

Def: Let $f: U \to \mathbb{R}$ be $C^2$ ($U \subseteq \mathbb{R}^n$ open),

then the Hessian matrix of $f$ at $\vec{a} \in U$ is

$$Hf(\vec{a}) = \begin{bmatrix} f_{x_1x_1}(\vec{a}) & \cdots & f_{x_1x_n}(\vec{a}) \\ \vdots & & \vdots \\ f_{x_nx_1}(\vec{a}) & \cdots & f_{x_nx_n}(\vec{a}) \end{bmatrix}$$

Remarks (1) $Hf(\vec{a})$ is $n \times n$ symmetric (by Clairaut's theorem)

(2) In textbook, Hessian of $f$ = $\det(Hf(\vec{a}))$

So we emphasize our definition is a matrix

eg: $f(x,y)$ at $(0,0)$

$$Hf(0,0) = \begin{bmatrix} f_{xx}(0,0) & f_{xy}(0,0) \\ f_{yx}(0,0) & f_{yy}(0,0) \end{bmatrix}$$

$$[x, y]\begin{bmatrix} f_{xx}(0,0) & f_{xy}(0,0) \\ f_{yx}(0,0) & f_{yy}(0,0)\end{bmatrix}\begin{bmatrix} x \\ y\end{bmatrix} = \underbrace{f_{xx}(0,0)\,x^2 + 2f_{xy}(0,0)\,xy + f_{yy}(0,0)\,y^2}$$

$2^{nd}$ order term in Taylor Polynominals (up to a factor $\frac{1}{2!}$)

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（答題不得寫在紅綫外）

$2^{nd}$ order Taylor polynominal of $f$ at $\vec{a}$ in matrix form

$$P_2(\vec{x}) = f(\vec{a}) + \vec{\nabla} f(\vec{a})(\vec{x}-\vec{a}) + \tfrac{1}{2}(\vec{x}-\vec{a})^T Hf(\vec{a})(\vec{x}-\vec{a}),$$

where $\vec{\nabla} f(\vec{a})$ regarded as row vector $[f_{x_1}(\vec{a}), \cdots, f_{x_n}(\vec{a})]$,

$\vec{x}-\vec{a}$ regarded as column vector $\begin{bmatrix} x_1-a_1 \\ \vdots \\ x_n-a_n\end{bmatrix}$

& $(\vec{x}-\vec{a})^T$ is the transpose $[x_1-a_1, \cdots, x_n-a_n]$.

eg $g(x,y) = \frac{\ln x}{1-y}$. Find $P_2(x,y)$ at $(1,0)$ using matrix form

$g(1,0) = 0$

$$\vec{\nabla} g = \left[\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right] = \left[\frac{1}{x(1-y)}, \frac{\ln x}{(1-y)^2}\right].$$

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$$Hg = \begin{bmatrix} \frac{\partial^2 g}{\partial x^2} & \frac{\partial^2 g}{\partial y \partial x} \\ \frac{\partial^2 g}{\partial x \partial y} & \frac{\partial^2 g}{\partial y^2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{x^2(1-y)} & \frac{1}{x(1-y)^2} \\ \frac{1}{x(1-y)^2} & \frac{2\ln x}{(1-y)^3} \end{bmatrix}$$

$$P_2(x,y) = g(1,0) + \vec{\nabla} g(1,0) \begin{bmatrix} x-1 \\ y \end{bmatrix}$$

$$= (x-1) - \frac{1}{2}(x-1)^2 + (x-1)y.$$

Application to local max/min

If $f$ is $C^2$, and $\vec{a}$ is a critical point of $f$,
then $\vec{\nabla} f(\vec{a}) = \vec{0}$.

$\Rightarrow f(\vec{x}) \simeq P_2(\vec{x}) = f(\vec{a}) + \vec{\nabla} f(a)(\vec{x}-\vec{a}) + \frac{1}{2}(\vec{x}-\vec{a})^T Hf(a)(\vec{x}-\vec{a})$

$\Rightarrow f(\vec{x}) - f(\vec{a}) \simeq \frac{1}{2}(\vec{x}-\vec{a})^T Hf(a)(\vec{x}-\vec{a})$

If $(\vec{x}-\vec{a})^T Hf(a)(\vec{x}-\vec{a}) < 0 \quad \forall \vec{x}$ near $\vec{a}$

then $f(\vec{x}) < f(\vec{a}) \quad \forall \vec{x}$ near $\vec{a}$

$\Rightarrow \vec{a}$ is a local max

If $(\vec{x}-\vec{a})^T Hf(a)(\vec{x}-\vec{a}) > 0 \quad \forall \vec{x}$ near $\vec{a}$

then $f(\vec{x}) > f(\vec{a}) \quad \forall \vec{x}$ near $\vec{a}$

$\Rightarrow \vec{a}$ is a local min

So we need to study when is a symmetric matrix $H$ satisfies

$\vec{v}^T H \vec{v} > 0 \quad \forall$ vector $\vec{v} \neq \vec{0}$

and $\vec{v}^T H \vec{v} < 0 \quad \forall$ vector $\vec{v} \neq \vec{0}$.

Hence we make the following

Def. Let $H$ be a symmetric $n \times n$ matrix. Then $H$ is said to be

(1) positive definite if $\vec{x}^T H \vec{x} > 0$

for all column vector $\vec{x} \in \mathbb{R}^n \setminus \{\vec{0}\}$

(2) negative definite if $\vec{x}^T H \vec{x} < 0$

for all column vector $\vec{x} \in \mathbb{R}^n \setminus \{\vec{0}\}$

(3) indefinite if $\exists$ column vectors $\vec{x}, \vec{y} \in \mathbb{R}^n \setminus \{\vec{0}\}$

such that $\vec{x}^T H \vec{x} > 0$ and $\vec{y}^T H \vec{y} < 0$.

Remark. These are not all possibilities: $\exists$ symmetric matrix which is not positive definite, negative definite, and indefinite

Then the discussion above implies

Thm (Second Derivative Test)

Let $\begin{cases} \cdot\ f: U \to \mathbb{R} \text{ be } C^2,\ U \subseteq \mathbb{R}^n \text{ open} \\ \cdot\ \vec{a} \in U \text{ such that } \vec{\nabla} f(\vec{a}) = \vec{0} \end{cases}$

Then

$Hf(\vec{a})$ is $\begin{cases} \text{positive definite} \Longrightarrow \vec{a} \text{ is a local min} \\ \text{negative definite} \Longrightarrow \vec{a} \text{ is a local max} \\ \text{indefinite} \Longrightarrow \vec{a} \text{ is a saddle point} \end{cases}$

Remark: A critical point which is neither local max nor local min is calleded a saddle point.

In particular for 2-variable, $\vec{v}^T H \vec{v}$ is of the function

$$g(x,y) = [x \quad y] \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = ax^2 + 2bxy + cy^2.$$

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eg (1) $[x,y]\begin{bmatrix}1&0\\0&4\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=x^2+4y^2>0,\ \forall\begin{bmatrix}x\\y\end{bmatrix}\in\mathbb{R}^2\setminus\{\vec{0}\}$

$\therefore\begin{bmatrix}1&0\\0&4\end{bmatrix}$ is positive definite

(2) $[x,y]\begin{bmatrix}-1&0\\0&-4\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=-x-4y^2<0,\ \forall\begin{bmatrix}x\\y\end{bmatrix}\in\mathbb{R}^2\setminus\{\vec{0}\}$

$\therefore\begin{bmatrix}-1&0\\0&-4\end{bmatrix}$ is negative definite

(3) $[x,y]\begin{bmatrix}-1&0\\0&4\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=-x^2+4y^2$

① If $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}$, then it is $4>0$.

② If $\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$, then it is $-1<0$.

$\therefore\begin{bmatrix}-1&0\\0&4\end{bmatrix}$ is indefinite.

(4) $[x,y]\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=x^2\geq 0$

But $[0\ 1]\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=0\ \cancel{>0}\Rightarrow$ not positive definite

$\therefore\begin{bmatrix}1&0\\0&0\end{bmatrix}$ is not positive definite, negative definite, not indefinite.

(5) $[x, y] \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x^2 + 4xy + 5y^2 > 0$

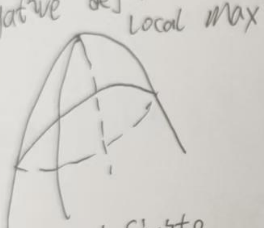
$\Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ is positive definite.

Geometrically (locally near the critical point)

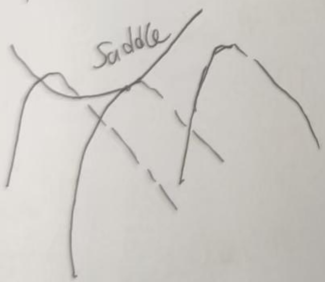
(1) $Hf(\vec{a})$ positive definite

The graph of $z = [x, y] Hf(\vec{a}) \begin{bmatrix} x \\ y \end{bmatrix}$ looks like


(2) $Hf(\vec{a})$ negative definite




(3) $Hf(\vec{a})$ is indefinite



Theorem Let $H = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$.

Then

$H$ is $\begin{cases} \text{positive definite} \Leftrightarrow \det H = ac - b^2 > 0, \ \& \ a > 0 \\ \text{negative definite} \Leftrightarrow \det H = ac - b^2 > 0, \ \& \ a < 0 \\ \text{indefinite} \Leftrightarrow \det H = ac - b^2 < 0 \end{cases}$

Pf: If $a \neq 0$, $[x\ y] \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = ax^2 + 2bxy + cy^2$

$= a(x^2 + 2\frac{b}{a}xy) + cy^2 = a\left[(x + \frac{b}{a}y)^2 + \frac{ac-b^2}{a^2}y^2\right]$

(1) $\det H = ac - b^2 > 0 \Leftrightarrow$

$[x\ y] \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ has the same sign as $a$

$\therefore$ The 1st two statements are proved.

(2) $\det H = ac - b^2 < 0 \Leftrightarrow$ 2 terms have different sign

$\Leftrightarrow \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ is indefinite

If $a = 0 \Rightarrow \det H = ac - b^2 = -b^2 \leq 0$

$[x\ y] \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2bxy + cy^2$

may be positive and may

eg 2: (NOT every function has global max/min)

(i) $f(x) = e^x$ on $\mathbb{R}$ (Domain unbounded) $\to$ No global max / No global min

(ii) $f(x) = x$ on $(-1, 1]$ (Domain not closed) $\to$ has global max

(iii) $f(x) = \begin{cases} 1-x, & 0 < x \leq 1, \\ 0, & x = 0, \\ -1-x, & -1 \leq x < 0. \end{cases}$

No global max and global min

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Extreme Value theorem

Let $\begin{cases} \bullet\ A \subseteq \mathbb{R}^n \text{ be closed and bounded,} \\ \bullet\ f: A \to \mathbb{R} \text{ be continuous} \end{cases}$

Then $f$ has global max and min

Proof: If $f$ is continuous on a bounded set, then we can know $f$ is bounded. we let $m$ and $M$ be the upper bound and lower bounded. (by least upper/lower bound of the real number). We also have there must be some $x \in A$, such that $M - f(x) < \varepsilon$. If no $x \in A$ such that $f(x) = m$ holds, we have $m - f(x) > 0$ always hold

$\Rightarrow \frac{1}{\varepsilon} > \frac{1}{m - f(x)}$, it is because $\frac{1}{m - f(x)}$ is still continuous in $A$, but we construct a sequence $\frac{1}{m - f(x)} \to \infty$, it is a contradiction.

Def: Let $\begin{cases} \cdot f: A \to \mathbb{R},\ A \in \mathbb{R}^n \text{ (not necessarily open)} \\ \cdot \vec{a} \in \mathrm{Int}(A) \end{cases}$

Then $\vec{a}$ is called a critical point of $f$ if
either (1) $\vec{\nabla} f(\vec{a})$ DNE,
or (2) $\vec{\nabla} f(\vec{a}) = \vec{0}$.

Theorem (First Derivative Test)
Suppose $f: A \to \mathbb{R}$ $(A \subset \mathbb{R}^n)$ attains a <u>local extreme</u> at $\vec{a} \in \mathrm{Int}(A)$, then $\vec{a}$ is a critical point of $f$.

proof: Suppose $f$ has a local extremum at $\vec{a} \in \mathrm{Int}(A)$, then $\vec{a}$ is a <u>critical point</u> of $f$. -

proof: Suppose $f$ has a local extremum at $\vec{a} \in \mathrm{Int}(A)$, If $\vec{\nabla} f(\vec{a})$ DNE, then $\vec{a}$ is a critical point. If $\vec{\nabla} f(\vec{a})$ exists, then $\frac{\partial f}{\partial x_i}(\vec{a})$ exists $\forall i = 1, \cdots, n$

$\forall i$, Consider the 1-variable function

$$g_i(t) = f(\vec{a} + t\hat{e}_i)$$

Where $g_i(0) = f(\vec{a})$ $\begin{matrix} \leq \\ \text{or} \\ \geq \end{matrix}$ $f(\vec{a} + t\hat{e}_i)$

$\Rightarrow$ $t = 0$ is a local max/min

$$\Rightarrow \left.\frac{dg_i}{dt}\right|_{t=0} = \left.\frac{df(\vec{a} + t\hat{e}_i)}{dt}\right|_{t=0} = \frac{\partial f}{\partial x_i}(\vec{a}) = 0$$

$$\lim_{t \to 0} \frac{f(\vec{a} + t\hat{e}_i) - f(\vec{a})}{t} \begin{cases} \geq 0 & \text{if } t > 0 \\ \leq 0 & \text{if } t < 0 \end{cases}$$

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Strategy for finding extreme

$f: A \to \mathbb{R}$

(1) Find critical points of $f$ in $Int(A)$.

(2) Study $f$ on boundary $\partial A$

find max\min of $f$ on $\partial A$.

(3) Compare values of $f$ at points found in steps (1) & (2).

第......题
（答题不得寫在紅綫外）

eg 1: Find global max\min of $f(x,y) = x^2+2y^2-x+3$ for $x^2+y^2 \le 1$

Solution: Step 1.

Critical points in $Int(A)$

$= Int\{x^2+y^2 \le 1\}$

$= \{x^2+y^2 < 1\}$

We know the polynomials always have derivatives, $\vec{\nabla} f = (2x-1, 4y)$.

$$\vec{0} = \vec{\nabla} f \iff \begin{cases} 2x-1=0 \\ 4y=0 \end{cases} \iff (x,y) = (\tfrac{1}{2}, 0)$$

$$f(\tfrac{1}{2}, 0) = \frac{11}{4}$$

Step 2: Study $f$ on $\partial\{x^2+y^2\le 1\} = \{x^2+y^2=1\}$

$$f(x,y) = x^2+2y^2-x+3 = 4+y^2-x$$
$$= 5-x^2-x$$
$$= 5-(x+\tfrac{1}{2})^2+\tfrac{1}{4}$$
$$= \tfrac{21}{4}-(x+\tfrac{1}{2})^2$$

max value of $f$ on $\partial A = \frac{21}{4}$

min value of $f$ on $\partial A$: $\frac{21}{4}-\frac{9}{4}=\frac{12}{4}=3$.



Step 3: Compare value of $f$ at points from steps (1) & (2).

$f(\frac{1}{2},0)=\frac{11}{4}$

$f(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2})=\frac{21}{4}$

$f(1,0)=3$

$$\Rightarrow \begin{cases} \text{max value} = \frac{21}{4} \\ \text{min value} = \frac{11}{4} \end{cases}$$