1.

Finding Local Extrema

Find all the local maxima, local minima, and saddle points of

(a)

$$f(x, y) = \sqrt{56x^2 - 8y^2 - 16x - 31} + 1 - 8x$$

(b)

$$f(x, y) = \frac{1}{x^2 + y^2 - 1}$$

(c)

$$f(x, y) = 2\ln x + \ln y - 4x - y$$

Solution:

(a) $$f_x(x, y) = \frac{56x - 8}{\sqrt{56x^2 - 8y^2 - 16x - 31}} - 8$$

$$f_y(x, y) = \frac{-8y}{\sqrt{56x^2 - 8y^2 - 16x - 31}}$$

Solving $f_x(x, y) = 0$ and $f_y(x, y) = 0$

yields $y = 0$ and $x = -2$ or $\frac{16}{7}$.

Therefore, the critical points are $(-2, 0)$ and $(\frac{16}{7}, 0)$

Then we calculate $f_{xx}(-2, 0) < 0$, $f_{xx}(\frac{16}{7}, 0) < 0$

$f_{yy}(-2, 0) < 0$ $f_{yy}(\frac{16}{7}, 0) < 0$

Also note that $f_{xy}(-2, 0) = f_{xy}(\frac{16}{7}, 0) = 0$

Applying the Second Derivative Test, $f_{xx} < 0$.
$f_{xx} f_{yy} - f_{xy}^2 > 0$. f has local maximum. at $(-2, 0)$ and $(\frac{16}{7}, 0)$.

(b) $f_x (x, y) = \frac{-2x}{(x^2+y^2-1)^2} = 0$ and $f_y = \frac{-2y}{(x^2+y^2-1)^2} = 0$

yielding the critical point $(0, 0)$.

$f_{xx} (x, y) = \frac{6x^2 - 2y^2 + 2}{(x^2+y^2-1)^3}$ $f_{yy} = \frac{6y^2 - 2x^2 + 2}{(x^2+y^2-1)^3}$

$f_{xy} = \frac{8xy}{(x^2+y^2-1)^3}$

$f_{xx} (0, 0) = -2$, $f_{yy} (0, 0) = -2$, $f_{xy} (0, 0) = 0$

By the Second Derivative Test,
$f_{xx} (0, 0) < 0$, $f_{xx} (0, 0) f_{yy} (0, 0) - f_{xy} (0, 0) = 4 > 0$.

f has local maximum $f(0, 0) = -1$.

(c). Solving
$f_x (x, y) = \frac{2}{x} - 4 = 0$ and $f_y (x, y) = \frac{1}{y} - 1 = 0$
yields $x = \frac{1}{2}$ and $y = 1$.

$f_{xx} (x, y) = -\frac{2}{x^2}$, $f_{yy} (x, y) = -\frac{1}{y^2}$, $f_{xy} (x, y) = 0$.
then $f_{xx} (\frac{1}{2}, 1) = -8 < 0$
$f_{xx} (\frac{1}{2}, 1) f_{yy} (\frac{1}{2}, 1) - f_{xy}^2 (\frac{1}{2}, 1) = 8 > 0$

Hence $f(x,y)$ has local maximum $f(\frac{1}{2},1) = -3-2\ln 2$.

2.

Show that $(0,0)$ is a critical point of $f(x, y) = x^2 + kxy + y^2$ no matter what value the constant k has. (*Hint:* Consider two cases: $k = 0$ and $k \neq 0$.)

Solution

If $k=0$, $f(x,y) = x^2+y^2$. Then $f_x(x,y) = 2x = 0$

$f_y(x,y) = 2y = 0$ gives $(0,0)$ is a critical point.

If $k \neq 0$, we have $f_x(x,y) = 2x+ky = 0$, then $y = -\frac{2x}{k}$

Substituting $y = -\frac{2x}{k}$ into $f_y(x,y) = 2y + kx = (-\frac{4}{k}+k)x = 0$

Then we obtain $x=0$ or $k = \pm 2$. It follows that $y=0$

or $y = \pm x$. Hence $(0,0)$ is a critical point.

3.

If $f_x(a, b) = f_y(a, b) = 0$, must f have a local maximum or minimum value at (a, b)? Give reasons for your answer.

Solution: NO.

Let $f(x,y) = xy$. then $f_x(x,y) = y = 0 \Rightarrow y=0$

and $f_y(x,y) = x = 0 \Rightarrow x = 0$.

It is clear that $(0,0)$ is a critical point and

$f_{xx}(0,0) = f_{yy}(0,0) = 0$, $f_{xy}(0,0) = 1$.

Therefore $f_{xx} f_{yy} - f_{xy}^2 = -1 < 0$.

which implies that $(0,0)$ is a saddle point.