Soln: Trick: $q(x,y,z) = \frac{1}{4}(x+y)^2 - \frac{1}{4}(x-y)^2 + z(x+y)$

Let $u = \frac{x+y}{2}$, $v = \frac{x-y}{2}$, then

$$
\begin{aligned}
q &= u^2 - v^2 + 2uz \\
&= (u^2 + 2uz + z^2) - v^2 - z^2 \\
&= (u+z)^2 - v^2 - z^2 \\
&= \frac{1}{4}(x+y+2z)^2 - \frac{1}{4}(x-y)^2 - z^2
\end{aligned}
$$

- On the plane $x+y+2z=0$ (i.e. $z = -\frac{x+y}{2}$)

$$q = q(x,y,-\tfrac{x+y}{2}) = -\frac{1}{4}(x-y)^2 - \frac{1}{4}(x+y)^2$$

$$< 0 \qquad \text{for } (x,y,-\tfrac{x+y}{2}) \neq \vec{0}$$

- Along the line $\begin{cases} x-y=0 \\ z=0 \end{cases} \Rightarrow \begin{cases} x=y \\ z=0 \end{cases}$

$$q = q(x,x,0) = \frac{1}{4}(x+x+0)^2 - 0^2 - 0^2 = x^2 > 0$$

$$(\forall\, x \neq 0,\ \text{i.e. } \forall\, (x,x,0) \neq \vec{0})$$

(Together $\Rightarrow$ $(0,0,0)$ is a saddle point) ※

Second Derivative Test for general n

Recall f is $C^2 \Rightarrow$ (by Clairaut's / mixed derivative Thm)

$$Hf(\vec{a}) = \left[f_{x_i x_j} \right]_{i,j=1,\cdots,n} \text{ is symmetric}$$

Theory of Linear Algebra $\Rightarrow$ Hf is diagonizable

i.e. $\exists$ orthogonal $n \times n$ matrix P (i.e $P^T P = Id$) s.t.

$$P^T Hf(\vec{a}) P = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

where λ_i, $i = 1, \cdots, n$, are eigenvalues of $Hf(\vec{a})$.

$$\Rightarrow \boxed{Hf(\vec{a}) \text{ is } \begin{cases} \text{positive definite} \Leftrightarrow \text{all } \lambda_i > 0 \\ \text{negative definite} \Leftrightarrow \text{all } \lambda_i < 0 \\ \text{indefinite} \Leftrightarrow \text{some } \lambda_i > 0, \text{ some } \lambda_j < 0 \text{ (all } \neq 0) \end{cases}}$$

Another way to check is consider determinants of submatrix

For each $1 \leq k \leq n$,
consider submatrix H_k given by
the upper left $k \times k$ entries.

$$\begin{bmatrix} f_{x_1 x_1} & \cdots & f_{x_1 x_k} & \cdots & f_{x_1 x_n} \\ \vdots & & \vdots & & \vdots \\ f_{x_k x_1} & \cdots & f_{x_k x_k} & \cdots & f_{x_k x_n} \\ \vdots & & \vdots & & \vdots \\ f_{x_n x_1} & \cdots & f_{x_n x_k} & \cdots & f_{x_n x_n} \end{bmatrix}$$

Then

$$\boxed{\begin{array}{l} Hf(\vec{a}) \text{ is positive definite} \Leftrightarrow \det H_k > 0, \ \forall k = 1, \cdots, n \\ Hf(\vec{a}) \text{ is negative definite} \Leftrightarrow \det H_k \begin{cases} < 0, & k \text{ odd} \\ > 0, & k \text{ even} \end{cases} \end{array}}$$

egs (1) $n=2$ $\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$ has $H_1 = [f_{xx}]$

$H_2 = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$

$\Rightarrow \quad \det H_1 = f_{xx}$

$\det H_2 = f_{xx} f_{yy} - f_{xy}^2$ (Same result as before)

(2) Diagonal matrix $\begin{bmatrix} \lambda_1 & & & & 0 \\ & \ddots & & & \\ & & \lambda_k & & \\ & & & \ddots & \\ 0 & & & & \lambda_n \end{bmatrix} \Rightarrow H_k = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_k \end{bmatrix}$

$\Rightarrow \det H_k = \lambda_1 \cdots \lambda_k$.